

Spectral Gaps of Quantum Hall Systems with Interactions

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A two-dimensional quantum Hall system without disorder for a wide class of interactions including any two-body interaction with finite range is studied by using the Lieb–Schultz–Mattis method [*Ann. Phys. (N.Y.)* **16**:407 (1961)]. The model is defined on an infinitely long strip with a fixed, large width, and the Hilbert space is restricted to the lowest $(n_{\max} + 1)$ Landau levels with a large integer n_{\max} . We prove that, for a noninteger filling ν of the Landau levels, either (i) there is a symmetry breaking at zero temperature or (ii) there is only one infinite-volume ground state with a gapless excitation. We also prove the following two theorems: (a) If a pure infinite-volume ground state has a nonzero excitation gap for a noninteger filling ν , then a translational symmetry breaking occurs at zero temperature. (b) Suppose that there is no non-translationally invariant infinite-volume ground state. Then, if a pure infinite-volume ground state has a nonzero excitation gap, the filling factor ν must be equal to a rational number. Here the ground state is allowed to have a periodic structure which is a consequence of the translational symmetry breaking. We also discuss the relation between our results and the quantized Hall conductance, and phenomenologically explain why odd denominators of filling fractions ν giving the quantized Hall conductance are favored exclusively.

KEY WORDS: Quantum Hall effect; fractional quantum Hall effect; Landau Hamiltonian; strong magnetic field; electron–electron interaction; spectral gap; translational symmetry breaking.

1. INTRODUCTION

Since the experimental discovery of the fractional quantum Hall effect,^(1,2) considerable theoretical efforts have been made to understand the nature of

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the ground state and of the low energy excitations above the ground state in the two-dimensional interacting electron gas in a strong magnetic field. Although there appeared many theories, mathematically rigorous or exact results are still fairly rare. Actually this is one of most difficult problems in solid state physics because the electron-electron interaction is essential to the fractional quantization of the Hall conductance. For the history of the quantum Hall effect, see refs. 3–5 and references therein.

In this paper we study the properties of infinite-volume ground states and of low energy excitations in a two-dimensional interacting electron gas in a uniform magnetic field without disorder for a wide class of electron-electron interactions. Although the class includes any two-body interaction with finite range, it does not include the standard Coulomb interaction proportional to $1/r$, where r is the distance between two electrons. Owing to technical reasons, we define the model on an infinitely long strip with a fixed width, and restrict the Hilbert space to the lowest $(n_{\max} + 1)$ Landau levels with a fixed integer n_{\max} . The precise form of the Hamiltonian is given in the next Section 2. In Section 2.5, the reason why we must fix the width and the integer n_{\max} , will be explained, and with the present results, we will give a discussion about the two-dimensional infinite-volume system with no restriction on the width and the cutoff n_{\max} .

We apply the Lieb–Schultz–Mattis method⁽⁶⁾ to the model. The method was developed to construct a low energy excitation above a finite-volume ground state for a lattice quantum spin system with a translational invariance. Later the method was applied to quantum spin chains in relation to the Haldane gap^(7,8) and magnetization plateaus.⁽⁹⁾ Yamanaka, Oshikawa and Affleck⁽¹⁰⁾ applied the method to a wide class of interacting fermions systems on a lattice.² Among these works, Oshikawa, Yamanaka and Affleck⁽⁹⁾ pointed out the analogy between the magnetization plateaus in a quantum spin chain and the conductance plateaus in the quantum Hall system. In both systems, a non-zero excitation gap above a ground state indeed plays an important role.

Using the Lieb–Schultz–Mattis method, an information about an infinite-volume ground state or a low energy excitation can be obtained for a translationally invariant system. In a quantum Hall system, it is believed that a non-zero excitation gap above a ground state leads to the quantization of the Hall conductance and the conductance plateaus. Therefore knowledge about a ground state and a low energy excitation is very important for the quantum Hall effect.

² The results of ref. 10 were revisited in specific cases by Gagliardini, Haas and Rice.⁽¹¹⁾

1.1. The Main Results of This Paper

Our results are as follows:

- Let the filling ν of the Landau levels be a non-integer. Then, either (i) there is a symmetry breaking at zero temperature or (ii) there is only one infinite-volume ground state with a gapless excitation.

- If a pure infinite-volume ground state has a non-zero excitation gap for a non-integer filling ν , then a translational symmetry breaking occurs at zero temperature.

- Suppose that there is no non-translationally invariant infinite-volume ground state. Then, if a pure infinite-volume ground state has a non-zero excitation gap, the filling factor ν must be equal to a rational number. Here the ground state is allowed to have a periodic structure which is a consequence of the translational symmetry breaking.

Here we stress that these statements hold also for a fixed macroscopic width of the strip and a fixed integer n_{\max} giving a macroscopic energy. But, in the proofs, the structure of the low energy excitation constructed by using the Lieb–Schultz–Mattis method strongly depends on the width of the strip and the energy cutoff n_{\max} . In particular, the size of the locally excited region must increase with increasing the cutoffs for keeping a small excitation energy. For this issue, we will give a discussion in Section 2.5. In the next Section 2, the above results will be given again as our main theorems in a mathematically rigorous manner. The mathematically precise definitions of the filling factor ν , an infinite-volume ground state and an excitation gap also will be given in the section.

1.2. Physical Meaning of the Results

Let us briefly discuss the physical meaning of the above our three results. To begin with, we remark the following: For an integer filling ν , a ground state has a trivial non-zero excitation gap³ which comes from the Landau levels for the non-interacting system if the magnetic field is sufficiently strong compared to the electron-electron interaction. We also remark that, without an interaction, there is no non-trivial structure leading to the fractional quantization of the Hall conductance. Thus we are interested in the case with a non-integer filling ν and with an interaction.

Since a non-trivial excitation gap above a ground state for a non-integer filling ν plays an important role for the fractional quantization of

³ It goes without saying that the integral quantization of the Hall conductance and the appearance of the conductance plateaus are non-trivial and suprising phenomena.^(12, 13)

the Hall conductance, the first case (i) in the first result is of interest to us. In this situation, a translational symmetry breaking occurs at zero temperature. This is the second result. In addition, if the electron-electron interaction is repulsive, we can expect that there is no non-translationally invariant ground state.⁴ But a pure infinite-volume ground state exhibits a periodic structure as a consequence of the translational symmetry breaking. Conversely, if the electron-electron interaction is attractive, we can expect that there is a phase separation which implies the existence of a non-translationally invariant ground state with no periodic structure. Thus, for the repulsive case, the assumption of the third result, i.e., the absence of non-translationally invariant ground states, is expected to be valid. With this assumption, the third result states that the filling factor ν must be equal to a rational number in the case of interest that there is a non-zero excitation gap above the ground state. Physically this implies that there appears a commensurate phase at zero temperature with a rational filling ν .

1.3. The Relation Between the Results and the Quantized Hall Conductance

Next we discuss the relation between the third result and the fractional quantization of the Hall conductance. The statements below in this subsection are not justified without additional assumptions to those of our present results.

To begin with, we briefly state our result about the Hall conductance in a separate paper.⁽¹⁴⁾ We treated a two-dimensional electrons gas in a uniform magnetic field for a wide class of potentials including single-body potentials with disorder and repulsive electron-electron interactions. We stress that there is a wide class of common models which are included in both the class of ref. 14 and that of the present paper. We obtained the following result: If there is a non-zero excitation gap above the ground state(s), then the Hall conductance σ_{xy} in the infinite-volume limit is given by⁵

$$\sigma_{xy} = -\frac{e^2}{h} \nu \quad (1.1)$$

⁴ As far as we know, there is no proof for this type of statement in the repulsive case.

⁵ As is well known, an argument relying on a topological invariant of the Hall conductance⁽¹⁵⁾ always yields an integral quantization of the Hall conductance without ad hoc assumptions.⁽¹⁶⁾ Since we did not rely on such an argument in our derivation of the Hall conductance, our result includes both integral and fractional quantizations of the Hall conductance. For earlier theoretical works on the Hall conductance, see refs. 17.

where $-e$ is the charge of electron, h is the Planck constant, and we assumed a regularization about a uniform electric field in the derivation of the Hall conductance. See ref. 14 for the mathematically rigorous statement. Unfortunately, the condition of a gap is different from that of the third result in the present paper, and we do not know the relation between the two conditions in a mathematically rigorous sense. In the rest of this subsection, we will use the conductance formula (1.1) without carefully examining the condition of a gap.

Let us consider a common model mentioned above, and make the assumptions for the third result in the present paper. Then we clearly have the fractional quantization of the Hall conductance by combining the rational filling ν of the third result with the conductance formula (1.1). Roughly speaking, a fractional filling factor ν with a non-zero excitation gap above a ground state gives the fractionally quantized Hall conductance. Next introduce weak disorder so that the non-zero excitation gap above the ground state in the clean system persists against disorder. Then we have the fractional quantization of the Hall conductance again because the conductance formula holds even for the presence of disorder.

The appearance of a Hall conductance plateau due to disorder will be discussed with relation to localization of wavefunctions in another separate paper.⁽¹⁸⁾

1.4. A Phenomenological Explanation for the Odd Denominator Rule

Experimental results show the surprising fact that odd denominators of filling fractions ν for which the quantization of the Hall conductance occurs, are favored exclusively. Namely non-zero excitation gaps appear only for filling fractions ν with odd denominators except for a few filling fractions with even denominators.⁽¹⁹⁾ Having our results in mind, we shall discuss the reason. Consider first the problem of two electrons with a repulsive interaction in a uniform magnetic field. Clearly the two electrons exert opposing forces on each other. But they cannot separate in the large distance because of the magnetic field. From this naive observation, one can expect that two electrons favor a bound pair⁽²⁰⁾ in a quantum Hall system.

Write $\nu = p/q$ with p, q mutually prime integers. Then there are p electrons and $q - p$ holes on q lattice sites, where the lattice is defined by an identification with the set of wavenumbers for the eigenvectors of the single-electron Landau Hamiltonian with the Landau gauge.⁶ Each

⁶ See Sections 2 and 3.1 for the precise definition of the lattice.

wavenumber is identical to the center of a harmonic oscillator part of an eigenvector. The set of all the centers is identical to the one-dimensional lattice. Assume q is an even integer. Then both p and $q - p$ are odd. This implies that neither the electrons nor the holes are grouped into bound pairs on the q lattice sites. To form a stable pairing state, we need $2q$ lattice sites which lead to a periodic structure with the period $2q$. Here the periodic structure is a consequence of a translational symmetry breaking. However, the filling $\nu = p/q$ is expected to lead to a structure with the period q , not $2q$. In consequence, we cannot expect a ground state with a non-zero excitation gap for an even denominator. Next assume q is odd. Then there are two possibilities: (i) p is odd and $q - p$ is even. (ii) p is even and $q - p$ is odd. Namely, either the number of the holes or the number of the electrons is even. Therefore either the holes or the electrons are grouped into bound pairs on the q lattice sites. In comparison to the case with an even denominator, we can expect a stable state, i.e., a ground state with a non-zero excitation gap. Unfortunately this is a phenomenological explanation which is still not justified.

1.5. Outline of This Paper

This paper is organized as follows: In Section 2, we give the precise definition of the model and some notions related to an infinite-volume ground state, and describe our main theorems in a mathematically rigorous manner. As preliminaries for the proofs of our theorems, we briefly review the eigenvalue problem of the single-electron Landau Hamiltonian and the degeneracy of finite-volume ground states in an interacting electrons system in Section 3. In Section 4, we construct a candidate for a low energy excitation above a ground state by using the Lieb–Schultz–Mattis method, and prove our main theorems. Section 5 is devoted to a proof of a proposition about the orthogonality between the excited and the ground states. The energy gaps are estimated in Section 6. For the convenience of readers, Appendices A–E are devoted to proofs of some technical theorem and lemmas.

2. THE MODEL AND THE MAIN THEOREMS

The purpose of this section is to describe our main theorems in a mathematically rigorous manner after giving mathematically precise definitions of an infinite-volume ground state and of a excitation gap for the quantum Hall system we consider.

2.1. The Hamiltonian

Consider a two-dimensional interacting electrons gas in a uniform magnetic field in a rectangular box $S := [-L_x/2, L_x/2] \times [-L_y/2, L_y/2]$. Although we consider electrons without spin degrees of freedom in this paper, our method is applied also to a quantum Hall system with spin degrees of freedom or with multiple layers.

The Hamiltonian of N electrons without spin degrees of freedom is given by

$$H^{(N)} = \sum_{j=1}^N \frac{1}{2m_e} [(p_{x,j} - eBy_j)^2 + p_{y,j}^2] + \sum_{j=1}^N W(x_j) + U^{(N)}(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) \quad (2.1)$$

where m_e and $-e$ are, respectively, the mass of electron and the charge of electron, and $(0, 0, B)$ is the uniform magnetic field perpendicular to the $x-y$ plane in which the electrons are confined; $\mathbf{r}_j = (x_j, y_j)$ is the j th Cartesian coordinate of the N electrons, and

$$p_{x,j} = -i\hbar \frac{\partial}{\partial x_j} \quad \text{and} \quad p_{y,j} = -i\hbar \frac{\partial}{\partial y_j} \quad (2.2)$$

with the Planck constant \hbar . The single-body potential W is a function of x only such that W is essentially bounded, i.e., $\|W\|_\infty < W_0 < \infty$ with a positive constant W_0 which is independent of L_x, L_y , and that W satisfies a periodic boundary condition as

$$W(x + L_x) = W(x) \quad \text{for any } x \in \mathbf{R} \quad (2.3)$$

A simple example of W is⁷

$$W(x) = W_0 \cos \kappa x \quad \text{with } \kappa = \frac{2\pi}{L_x} n, \quad n \in \mathbf{Z} \quad (2.4)$$

where W_0 is a real constant. The interaction $U^{(N)}$ is written in a sum of two-body interactions as

$$U^{(N)}(\mathbf{r}_1, \dots, \mathbf{r}_N) = \sum_{i < j} U^{(2)}(x_i - x_j, y_i - y_j) \quad (2.5)$$

⁷The question of the applicability of our method to a quantum Hall system with a periodic potential was brought to the author by Mahito Kohmoto. Thus we have partially answered his question, although we still cannot treat a periodic potential modulating in both x and y directions.

The two-body interaction $U^{(2)}$ is invariant under the exchange of two coordinates of the electrons, i.e.,

$$U^{(2)}(-x, -y) = U^{(2)}(x, y) \quad (2.6)$$

We assume that $U^{(2)}$ satisfies the periodic boundary conditions

$$U^{(2)}(x + L_x, y) = U^{(2)}(x, y + L_y) = U^{(2)}(x, y) \quad (2.7)$$

Further we assume that $U^{(2)}$ is continuous on \mathbf{R}^2 , and satisfies

$$|U^{(2)}(x, y)| \leq U_0 \{1 + [\text{dist}(x, y)/r_0]^2\}^{-\gamma/2} \quad \text{for } (x, y) \in \mathbf{R}^2 \quad (2.8)$$

with $\gamma > 2$ and with positive constants U_0, r_0 . Here the distance is given by

$$\text{dist}(x, y) := \sqrt{\min_{m \in \mathbf{Z}} \{|x - mL_x|^2\} + \min_{n \in \mathbf{Z}} \{|y - nL_y|^2\}} \quad (2.9)$$

We take $L_x L_y = 2\pi M \ell_B^2$ with a sufficiently large positive integer M . Here ℓ_B is the magnetic length, i.e., $\ell_B := \sqrt{\hbar/eB}$. For simplicity, we take M even. This condition for L_x, L_y is convenient for imposing periodic boundary conditions as follows: For an N -electron wavefunction $\Phi^{(N)}$, we impose periodic boundary conditions

$$t_j^{(x)}(L_x) \Phi^{(N)}(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = \Phi^{(N)}(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) \quad (2.10)$$

and

$$t_j^{(y)}(L_y) \Phi^{(N)}(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = \Phi^{(N)}(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) \quad (2.11)$$

for $j=1, 2, \dots, N$. Here $t^{(x)}(\dots)$ and $t^{(y)}(\dots)$ are magnetic translation operators⁽²¹⁾ defined as

$$\begin{aligned} t^{(x)}(x') f(x, y) &= f(x - x', y) \\ t^{(y)}(y') f(x, y) &= \exp[iy'x/\ell_B^2] f(x, y - y') \end{aligned} \quad (2.12)$$

for a function f on \mathbf{R}^2 , and a subscript j of an operator indicates that the operator acts on the j th coordinate of a function.⁸ The ranges of x' and y' are given by (see Section 3.1)

$$x' = m \Delta x \quad \text{with } m \in \mathbf{Z} \quad \text{and} \quad y' = n \Delta y \quad \text{with } n \in \mathbf{Z} \quad (2.13)$$

⁸ Throughout the present paper we use this convention.

where the minimal units of the translations are given by

$$\Delta x := \frac{\hbar}{eB} \frac{1}{L_y} \quad \text{and} \quad \Delta y := \frac{\hbar}{eB} \frac{1}{L_x} \quad (2.14)$$

Owing to certain technical reasons,⁹ we must restrict the whole Hilbert space to the lowest $(n_{\max} + 1)$ Landau levels with a large positive integer n_{\max} . In order to give a more precise definition of the restriction, consider the Hamiltonian of a single electron given by

$$\mathcal{H} = \frac{1}{2m_e} [(p_x - eBy)^2 + p_y^2] \quad (2.15)$$

with periodic boundary conditions

$$\phi(x, y) = t^{(x)}(L_x) \phi(x, y), \quad \phi(x, y) = t^{(y)}(L_y) \phi(x, y) \quad (2.16)$$

for the wavefunction ϕ , with $L_x L_y = 2\pi M \ell_B^2$ with M even. The explicit forms of the normalized eigenvectors $\phi_{n,k}^P$ of the Hamiltonian \mathcal{H} are given in Section 3.1. Here $n \in \{0, 1, 2, \dots\}$ is a Landau index, and k is a wavenumber given by $k = 2\pi m/L_x$ with $m \in \Lambda^{(M)} = \{-M/2 + 1, -M/2 + 2, \dots, M/2\}$. The energy eigenvalue is given by

$$\mathcal{E}_{n,k} := (n + \frac{1}{2}) \hbar \omega_c \quad (2.17)$$

with $\omega_c := eB/m_e$. The system $\{\phi_{n,k}^P\}_{n,k}$ is the orthonormal complete system.

Now we define the restriction of the Hilbert space, i.e., the energy cutoff. For a nonnegative integer n_{\max} , we define by $P(n_{\max})$ the spectral projection onto the subspace spanned by all the eigenvectors with the Landau indices $n \leq n_{\max}$. Namely, by the projection $P(n_{\max})$, the whole Hilbert space is restricted to the lowest $(n_{\max} + 1)$ Landau levels. The corresponding N electrons Hamiltonian is given by

$$H^{(N)}(n_{\max}) = P^{(N)}(n_{\max}) H^{(N)} P^{(N)}(n_{\max}) \quad (2.18)$$

with the projection

$$P^{(N)}(n_{\max}) := \bigotimes_{j=1}^N P_j(n_{\max}) \quad (2.19)$$

⁹ See Section 2.5 for the detail.

2.2. A C^* Algebraic Approach

Throughout the present paper, we consider the thermodynamic limit $L_y \rightarrow \infty$ for a fixed L_x and a fixed n_{\max} . Namely we consider an infinitely long strip with a finite width¹⁰ L_x . In this limit, we also fix the filling factor ν which is given by $\nu = N/M$ for a finite volume with $L_x L_y = 2\pi M \ell_B^2$. For treating the infinite-volume system, it is convenient to introduce the notion of local observables by following the idea of a C^* algebra.⁽²²⁾ Although a C^* algebra must be a fairly mathematical tool, it enables us to avoid confusion between the degeneracy of finite-volume ground states and that of infinite-volume ground states.⁽²³⁾ In addition it clarifies the notions of low energy excitations and of a gap above an infinite-volume ground state.

In order to introduce the notion of local observables, we first consider the second quantized form of the Hamiltonian 2.18). It is written as¹¹

$$\begin{aligned}
 H_{A^{(M)}}(n_{\max}) := & \sum_{n=0}^{n_{\max}} \sum_{m \in A^{(M)}} (n + \frac{1}{2}) \hbar \omega_c c_{n,m}^* c_{n,m} \\
 & + \sum_{j,\alpha} \sum_{j',\alpha'} W(j,\alpha; j',\alpha') c_{j,\alpha}^* c_{j',\alpha'} \\
 & + \frac{1}{2} \sum_{j,\alpha} \sum_{\ell,\beta} \sum_{j',\alpha'} \sum_{\ell',\beta'} U^{(2)}(j,\alpha; \ell,\beta; j',\alpha'; \ell',\beta') \\
 & \times c_{j,\alpha}^* c_{\ell,\beta}^* c_{j',\alpha'} c_{\ell',\beta'} \quad (2.20)
 \end{aligned}$$

with

$$W(j,\alpha; j',\alpha') := \int_S dx dy [\phi_{j,p}^P(x,y)]^* W(x) \phi_{j',p'}^P(x,y) \quad (2.21)$$

and

$$\begin{aligned}
 & U^{(2)}(j,\alpha; \ell,\beta; j',\alpha'; \ell',\beta') \\
 & := \int_S dx dy \int_S dx' dy' [\phi_{j,p}^P(x,y)]^* [\phi_{\ell,q}^P(x',y')]^* \\
 & \quad \times U^{(2)}(x-x', y-y') \phi_{j',p'}^P(x',y') \phi_{\ell',q'}^P(x,y) \quad (2.22)
 \end{aligned}$$

¹⁰ The reason why we must fix the width to finite will be explained in Section 2.5.

¹¹ A perturbative approach starting from the Hamiltonian (2.20) was treated in refs. 24. For other approximate methods, see refs. 25.

Here we have written

$$p = 2\pi\alpha/L_x, \quad q = 2\pi\beta/L_x, \quad p' = 2\pi\alpha'/L_x, \quad q' = 2\pi\beta'/L_x \quad (2.23)$$

and $c_{n,m}$ and $c_{n,m}^*$ are, respectively, the electron annihilation and creation operators for the eigenstate $\phi_{n,k}^P$ of the single electron Landau Hamiltonian \mathcal{H} of (2.15) with the wavenumber $k = 2\pi m/L_x$. These annihilation and creation operators satisfy the canonical anticommutation relations as

$$\{c_{n,m}, c_{n',m'}\} = 0, \quad \{c_{n,m}, c_{n',m'}^*\} = \delta_{n,n'} \delta_{m,m'} \quad (2.24)$$

We can identify the quantum number $m \in A^{(M)}$ with the lattice site m in the one-dimensional lattice $A^{(M)} = \{-M/2 + 1, -M/2 + 2, \dots, M/2\}$. In other words, the set of all the wavenumbers k is identical to the one-dimensional lattice. A wavenumber k is corresponding to the center of the harmonic oscillator part of the wavefunction¹² $\phi_{n,k}^P$. From this identification, the present system of the Hamiltonian $H_{A^{(M)}(n_{\max})}$ of (2.20) is identical to a one-dimensional lattice fermions system with long-range interactions and without spin degrees of freedom. Then the original Landau levels with a wavenumber $k = 2\pi m/L_x$ are interpreted as atomic levels at the corresponding lattice site m .

We note that the electron–electron interaction $U^{(2)}$ of the present system satisfies the condition

$$\lim_{A^{(M)} \uparrow \mathbf{Z}} \max_{j, \alpha} \sum_{\ell, \beta} \sum_{j', \alpha'} \sum_{\ell', \beta'} |U^{(2)}(j, \alpha; \ell, \beta; j', \alpha'; \ell', \beta')| < \infty \quad (2.25)$$

One can easily prove this condition by using Lemmas 6.3 and 6.4 below. By this condition, the total energy of a finite volume is of order of the volume. Further, the condition guarantees the existence of the time evolution of a local observable. Roughly speaking, the condition is equivalent to

$$\int_{\mathbf{R}^2} dx dy |U^{(2)}(x, y)| < \infty \quad (2.26)$$

Clearly this condition is too strong. In fact, the standard Coulomb interaction does not satisfy the condition.

Since the operator $c_{n,m}^*$ creates the single electron wavefunction $\phi_{n,k}^P$ in the $L_x \times L_y$ rectangular box in the Fock space, the annihilation and creation operators $c_{n,m}, c_{n,m}^*$ depend on the system size L_y . This fact is not convenient for introducing local observables in the following because it is

¹² See Section 3.1.

very hard to treat the outside of the rectangular box with the operators $c_{n,m}, c_{n,m}^*$. In order to avoid this difficulty, we introduce different abstract annihilation and creation operators $\tilde{c}_{n,m}, \tilde{c}_{n,m}^*$ with $m \in \mathbf{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$. These operators also obey the canonical anti-commutation relations

$$\{\tilde{c}_{n,m}, \tilde{c}_{n',m'}\} = 0, \quad \{\tilde{c}_{n,m}, \tilde{c}_{n',m'}^*\} = \delta_{n,n'} \delta_{m,m'} \quad (2.27)$$

Namely $\tilde{c}_{n,m}, \tilde{c}_{n,m}^*$ are defined on the infinite lattice \mathbf{Z} . We replace $c_{n,m}, c_{n,m}^*$ with $\tilde{c}_{n,m}, \tilde{c}_{n,m}^*$ in the Hamiltonian (2.20). As a result, we have the Hamiltonian

$$\begin{aligned} \tilde{H}_{A^{(M)}}(n_{\max}) &:= \sum_{n=0}^{n_{\max}} \sum_{m \in A^{(M)}} (n + \frac{1}{2}) h\omega_c \tilde{c}_{n,m}^* \tilde{c}_{n,m} \\ &+ \sum_{j,\alpha} \sum_{j',\alpha'} W(j,\alpha : j',\alpha') \tilde{c}_{j,\alpha}^* \tilde{c}_{j',\alpha'} \\ &+ \frac{1}{2} \sum_{j,\alpha} \sum_{\ell,\beta} \sum_{j',\alpha'} \sum_{\ell',\beta'} U^{(2)}(j,\alpha; \ell,\beta : j',\alpha'; \ell',\beta') \\ &\times \tilde{c}_{j,\alpha}^* \tilde{c}_{\ell,\beta}^* \tilde{c}_{j',\alpha'} \tilde{c}_{\ell',\beta'} \end{aligned} \quad (2.28)$$

with the same periodic boundary conditions on the same finite lattice $A^{(M)}$ as in $H_{A^{(M)}}(n_{\max})$. Clearly $\tilde{H}_{A^{(M)}}(n_{\max})$ has the same spectrum as that of $H_{A^{(M)}}(n_{\max})$.

Now we introduce local observables. Let A be a subset of \mathbf{Z} . We denote by \mathcal{A}_A the set of all the observables generated by all the annihilation $\tilde{c}_{n,m}$ and the creation $\tilde{c}_{n',m'}^*$ operators with $m, m' \in A$ and with $n, n' \in \{0, 1, \dots, n_{\max}\}$. We define the set of the local observables \mathcal{A}_{loc} as

$$\mathcal{A}_{\text{loc}} := \bigcup_{A \subset \mathbf{Z}; |A| < \infty} \mathcal{A}_A \quad (2.29)$$

Let A^c be the complement of the lattice A , i.e., $A^c = \mathbf{Z} \setminus A$. Then \mathcal{A}_{A^c} is the algebra for the outside of A . Roughly speaking, the algebra \mathcal{A}_{A^c} is an algebra for the outside of the $L_x \times L_y$ rectangular box because an original wavenumber $k = 2\pi m/L_x$ is identical to the center of the harmonic oscillator part of the wavefunction $\phi_{n,k}^{\text{P}}$.

Next we introduce a set of $U(1)$ global gauge transformations. A global gauge transformation U_θ in the set is defined as

$$U_\theta(\tilde{c}_{n,m}) = e^{-i\theta} \tilde{c}_{n,m}, \quad U_\theta(\tilde{c}_{n,m}^*) = e^{i\theta} \tilde{c}_{n,m}^* \quad (2.30)$$

with $\theta \in [0, 2\pi)$. Namely a $U(1)$ gauge transformation U_θ is a global phase twist with a real angle θ for the quantum mechanical phase of wavefunctions. Following Matsui,⁽²⁶⁾ we define by $\mathcal{A}_{\text{loc}}^{U(1)}$ the $U(1)$ gauge invariant part of \mathcal{A}_{loc} , i.e.,

$$\mathcal{A}_{\text{loc}}^{U(1)} := \{a \in \mathcal{A}_{\text{loc}} \mid U_\theta(a) = a \text{ for all } \theta \in [0, 2\pi)\} \quad (2.31)$$

2.3. Infinite-Volume Ground States and Excitation Gaps

Let $\tilde{\Phi}_{\mathcal{A}^{(M)}}^{(N)}$ be a normalized N electrons ground state of the Hamiltonian $\tilde{H}_{\mathcal{A}^{(M)}}^{(N)}(n_{\text{max}})$ of (2.28). Clearly $\tilde{\Phi}_{\mathcal{A}^{(M)}}^{(N)}$ is identical to a ground state $\Phi_{L_y}^{(N)}$ of the Hamiltonian $H_{L_y}^{(N)}(n_{\text{max}}) := H^{(N)}(n_{\text{max}})$ of (2.18) with the system size L_y in the y direction. Then an infinite-volume ground state ω can be constructed as¹³

$$\omega(a) = \lim_{\mathcal{A}^{(M)} \uparrow \mathbf{Z}} \langle \tilde{\Phi}_{\mathcal{A}^{(M)}}^{(N)}, a \tilde{\Phi}_{\mathcal{A}^{(M)}}^{(N)} \rangle \quad (2.32)$$

for a local observable $a \in \mathcal{A}_{\text{loc}}^{U(1)}$, and for fixed L_x , n_{max} and v . All the infinite-volume ground states thus obtained are not necessarily complete as physically natural ground states. See ref. 27 for example. We use a more general definition of infinite-volume ground states as follows: A state ω , i.e., a positive normalized linear functional, on local observables $\mathcal{A}_{\text{loc}}^{U(1)}$ is an infinite-volume ground state if and only if ω satisfies the local stability condition¹⁴

$$\lim_{\mathcal{A} \uparrow \mathbf{Z}} \omega(a^*[\tilde{H}_{\mathcal{A}}(n_{\text{max}}), a]) \geq 0 \quad (2.33)$$

for any local observable $a \in \mathcal{A}_{\text{loc}}^{U(1)}$. From the definition of the vector $\tilde{\Phi}_{\mathcal{A}^{(M)}}^{(N)}$, the infinite-volume ground state (2.32) satisfies the condition (2.33) as

$$\begin{aligned} & \lim_{\mathcal{A} \uparrow \mathbf{Z}} \omega(a^*[\tilde{H}_{\mathcal{A}}(n_{\text{max}}), a]) \\ &= \lim_{\mathcal{A}^{(M)} \uparrow \mathbf{Z}} \langle \tilde{\Phi}_{\mathcal{A}^{(M)}}^{(N)}, a^*[\tilde{H}_{\mathcal{A}^{(M)}}(n_{\text{max}}) - E_{L_y}^{(N)}] a \tilde{\Phi}_{\mathcal{A}^{(M)}}^{(N)} \rangle \\ &= \lim_{L_y \uparrow \infty} \langle \Phi_{L_y}^{(N)}, \hat{a}^*[H_{L_y}^{(N)}(n_{\text{max}}) - E_{L_y}^{(N)}] \hat{a} \Phi_{L_y}^{(N)} \rangle \geq 0 \end{aligned} \quad (2.34)$$

for $a \in \mathcal{A}_{\text{loc}}^{U(1)}$. Here $E_{L_y}^{(N)}$ is the energy eigenvalue of $\Phi_{L_y}^{(N)}$ for the Hamiltonian $H_{L_y}^{(N)}(n_{\text{max}})$ of (2.18), and \hat{a} is the observable corresponding to the observable a .

¹³ If necessary, we take a subsequence for the limit $\mathcal{A}^{(M)} \uparrow \mathbf{Z}$.

¹⁴ For more details, see ref. 22.

We denote by $\tau_j^{(y)}$ the shift operator by j lattice sites in the y direction. Namely the shift operator is defined as

$$\tau_j^{(y)}(\tilde{c}_{n,m}) = \tilde{c}_{n,m+j}, \quad \tau_j^{(y)}(\tilde{c}_{n,m}^*) = \tilde{c}_{n,m+j}^* \quad (2.35)$$

Let ω be an infinite-volume ground state. We say that ω is translationally invariant with a period $q \in \mathbf{N} := \{1, 2, \dots\}$ if and only if ω satisfies

$$\omega(\tau_q^{(y)}(\dots)) = \omega(\dots) \quad (2.36)$$

If a ground state ω has a non-trivial minimal period $q \neq 1$, then a translational symmetry breaking occurs at zero temperature. If a ground state ω has no period, then we say that ω is a non-translationally invariant ground state.

Consider the Hamiltonian with a chemical potential μ ,

$$\tilde{H}_{A,\mu}(n_{\max}) := \tilde{H}_A(n_{\max}) - \mu \sum_{n=0}^{n_{\max}} \sum_{m \in A} \tilde{n}_{n,m} \quad (2.37)$$

with the electron number operator

$$\tilde{n}_{n,m} := \tilde{c}_{n,m}^* \tilde{c}_{n,m} \quad (2.38)$$

For the grand-canonical ensemble, the definition of an infinite-volume ground state is given as follows: A state ω is an infinite-volume ground state if and only if

$$\lim_{A \uparrow \mathbf{Z}} \omega(a^*[\tilde{H}_{A,\mu}(n_{\max}), a]) \geq 0 \quad (2.39)$$

for any $a \in \mathcal{A}_{\text{loc}}$. Matsui⁽²⁶⁾ proved an equivalence between a canonical ensemble and a grand-canonical ensemble for a lattice fermion system with a certain interaction. The following theorem for the present quantum Hall system follows from the Matsui's result.

Theorem 2.1. Let ω be a translationally invariant infinite-volume ground state with a period for $\mathcal{A}_{\text{loc}}^{U(1)}$. Then there exists a chemical potential μ such that the gauge invariant extension $\tilde{\omega}$ of ω to \mathcal{A}_{loc} is an infinite-volume ground state for \mathcal{A}_{loc} .

A sketch of the proof is given in Appendix A.

Next we shall introduce a definition of a gap above an infinite-volume ground state. For this purpose, we first define the time evolution of a local observable $a \in \mathcal{A}_{\text{loc}}$ as

$$\tau_{t, A}(a) := \exp[i\tilde{H}_{A, \mu}(n_{\max}) t/\hbar] a \exp[-i\tilde{H}_{A, \mu}(n_{\max}) t/\hbar] \quad (2.40)$$

and its infinite-volume limit,

$$\tau_t(a) := \lim_{A \uparrow \mathbf{Z}} \tau_{t, A}(a) \quad (2.41)$$

In the sense of the norm, this limit exists uniformly for time t in a compact set. Let \mathcal{A} be the norm completion of \mathcal{A}_{loc} . Then $\tau_t(a)$ is defined also for $a \in \mathcal{A}$. Further we define

$$\tau_{*f}(a) := \int_{-\infty}^{+\infty} dt f(t) \tau_t(a) \quad (2.42)$$

for a function f on \mathbf{R} and $a \in \mathcal{A}$ when the right-hand side exists. We denote by C_0^∞ the set of infinitely differentiable functions with compact support.

Definition 2.2. An infinite-volume ground state ω has a gap γ if and only if the following condition is satisfied: Let f be a function on \mathbf{R} with Fourier transform $\hat{f} \in C_0^\infty$ and $\text{supp } \hat{f} \subseteq (0, \gamma)$, then

$$\omega([\tau_{*f}(a)]^* \tau_{*f}(a)) = 0 \quad (2.43)$$

for all $a \in \mathcal{A}$.

This definition of a gap is slightly different from that in ref. 7. For the gauge invariant extension $\tilde{\omega}$ of ω of (2.32), the left-hand side of (2.43) becomes

$$\begin{aligned} & \tilde{\omega}([\tau_{*f}(a)]^* \tau_{*f}(a)) \\ &= \lim_{A \uparrow \mathbf{Z}} \langle \tilde{\Phi}_A^{(N)}, a^* [\hat{f}(\{\tilde{H}_{A, \mu}(n_{\max}) - E_{L_y}^{(N)} + \mu N\}/\hbar)]^2 a \tilde{\Phi}_A^{(N)} \rangle \end{aligned} \quad (2.44)$$

Thus the above definition of a gap above an infinite-volume ground state is a physically natural definition for the states ω of (2.32). In particular, the gap condition (2.43) becomes

$$\begin{aligned} & \lim_{A \uparrow \mathbf{Z}} \langle \tilde{\Phi}_A^{(N)}, a^* [\hat{f}(\{\tilde{H}_A(n_{\max}) - E_{L_y}^{(N)}\}/\hbar)]^2 a \tilde{\Phi}_A^{(N)} \rangle \\ &= \lim_{L_y \rightarrow \infty} \langle \Phi_{L_y}^{(N)}, \hat{a}^* [\hat{f}(\{H_{L_y}(n_{\max}) - E_{L_y}^{(N)}\}/\hbar)]^2 \hat{a} \Phi_{L_y}^{(N)} \rangle = 0 \end{aligned} \quad (2.45)$$

for $a \in \mathcal{A}_{\text{loc}}^{U(1)}$. Here \hat{a} is the observable corresponding to the observable a . We remark that ω is an infinite-volume ground state for \mathcal{A}_{loc} if and only if the following condition is satisfied: Let f be a function on \mathbf{R} with Fourier transform $\hat{f} \in C_0^\infty$ and $\text{supp } \hat{f} \subseteq (-\infty, 0)$, then

$$\omega([\tau_{*f}(a)]^* \tau_{*f}(a)) = 0 \quad (2.46)$$

for all $a \in \mathcal{A}$. See ref. 22 for the detail.

2.4. Main Theorems of This Paper

Now we describe our main theorems. In the following, we fix the width L_x of the strip and the energy cutoff n_{max} to finite values.

Theorem 2.3. Suppose the filling factor ν is not an integer. Then, either (i) there is more than one infinite-volume ground state or (ii) there is only one infinite-volume ground state with a gapless excitation.

In the case (i), there is a symmetry breaking at zero temperature. Since a non-zero excitation gap plays an important role for the quantization of the Hall conductance in a quantum Hall system, we are not interested in the case (ii).

Theorem 2.4. Suppose that the filling factor ν is not an integer and that a pure infinite-volume ground state has a non-zero excitation gap. Then a translational symmetry breaking occurs at zero temperature.

Thus a translational symmetry breaking inevitably occurs in the situation where there appears a fractional quantization of the Hall conductance which is observed with a non-zero excitation gap for a fractional filling. In a realistic situation where the electron-electron interaction is repulsive, we can expect that there is no non-translationally invariant ground state with no periodic structure as we mentioned in Section 1.2.

Theorem 2.5. Suppose that there is no non-translationally invariant infinite-volume ground state. Then, if a pure infinite-volume ground state ω has a non-zero excitation gap, the filling factor ν must be equal to a rational number. In particular, if the ground state has a periodic structure with a minimal period $q \in \mathbf{N}$ for the magnetic translation in the y direction, the filling factor ν must satisfy $q\nu \in \mathbf{N}$.

Here, if the period q is equal to the denominator of the filling ν as in a usual commensurate phase, we have $\nu = p/q$ with p, q mutually prime integers. The relation between Theorem 2.5 and the fractional quantization

of the Hall conductance was already discussed with the results of a separate paper⁽¹⁴⁾ in Section 1.3. The appearance of the Hall conductance plateau will be discussed with relation to localization of wavefunctions in another separate paper.⁽¹⁸⁾

2.5. The Finite Width of the Strip and the Energy Cutoff n_{\max}

In the above we have fixed the width L_x of the strip and the energy cutoff n_{\max} to finite values. Although the statements of our three theorems hold even for a fixed macroscopic width and for a fixed n_{\max} giving a macroscopic energy cutoff, the structure of the low energy excitation constructed by using the Lieb–Schultz–Mattis method strongly depends on these cutoffs. In particular, the size of the locally excited region must increase with increasing the cutoffs for keeping a small excitation energy. Before concluding this section, we shall give discussions about this cutoff dependence of the excitation and about the two-dimensional infinite-volume system with no such restrictions.

Consider first the energy cutoff n_{\max} . We recall the model described by the Hamiltonian (2.28). The model is identical to a one-dimensional lattice fermion system with long-range interactions. The range of the interactions strongly depends on the cutoff n_{\max} . Actually the effective range seems to increase with increasing the energy of a fermion state. As a result, the upper bound of the excitation energy of the state constructed by using the Lieb–Schultz–Mattis method depends on the cutoff n_{\max} and is divergent as n_{\max} tends to infinity. For the explicit cutoff dependence,¹⁵ see Section 6. Although we need an infinitesimally small upper bound of the excitation energy for a large volume, we can not get a desired one without the cutoff. This is nothing but the reason why we introduced the cutoff n_{\max} into the Hilbert space. However, one can expect generally that the contribution of very high energy states to low energy quantities is negligibly small. In fact, the energy of the excitation constructed by the Lieb–Schultz–Mattis method can be written in the ground state expectation of an operator. (See Section 4 for the detail.) Clearly the difference between the ground state expectation with the cutoff and that without the cutoff is determined by the high energy states which are cut off. If the contribution of the high energy states is negligibly small, then the upper bound of the energy of the excitation thus constructed is independent of the cutoff n_{\max} , and we can remove the cutoff. Unfortunately we could not get a useful estimate for the contribution of the high energy states.

¹⁵ The cutoff n_{\max} dependence of the energy bound is too complicated to be written explicitly here.

Next we give a discussion about the cutoff L_x of the width of the strip. In order to prove our main three theorems, we construct a low energy excitation above a ground state by relying on the Lieb–Schultz–Mattis method. Here we stress that locality of the excitation is absolutely essential for the proofs. However, the constructed excitation is extended homogeneously from end to end in the x direction. (See Section 4 for the detail.) Moreover, in the y direction it has a linear size δy which strongly depends on the width L_x as

$$\delta y \propto L_x^{3+\varepsilon} \quad (2.47)$$

Here ε is a positive small number. For the detail, see Section 6.1.2. In order to treat the two-dimensional infinite-volume system with no such a cutoff in this approach, we need to construct a low energy excitation state which is local in both x and y directions. Unfortunately we could not construct such a low energy state, and we fixed the width L_x to a finite value. In order to overcome this difficulty, it seems to us that a new idea beyond the Lieb–Schultz–Mattis method is required.

Although we failed to overcome the difficulty, we can give a physically plausible argument to show the existence of a low energy excitation which is local in both x and y directions. To begin with, we note the following folk statement which is not generally justified, but physically plausible: If a system with a volume has a low energy excitation, then the same system with a larger volume also has a similar excitation in the sense that the corresponding excitation in the larger system keeps the same orders of the spatial extent and the excitation energy as those of the small system. Having this folk statement in mind, let us consider the two quantum Hall systems of infinitely long strips with the widths L_x and $L'_x \gg L_x$. Fix L_x . Then we have an excitation with a low energy ΔE and with the linear size δy in the y direction and L_x in the x direction, following the Lieb–Schultz–Mattis method. Here, if the above folk statement is true, we have a local excitation with a low energy of the same order ΔE and with the linear size of order δy in the y direction and of order L_x in the x direction for the system with the large width L'_x . Thus we can expect the existence of a low energy excitation which is local in both x and y directions. However, it is not so easy to construct such an excitation. In fact, we could not construct it.

In conclusion, we believe that our three results hold also for the two-dimensional infinite-volume quantum Hall system without the energy and the spatial cutoffs n_{\max} , L_x , and that these conjectures will be justified in future studies.

3. PRELIMINARIES

As preliminaries for the proofs of our main theorems, we briefly review the eigenvalue problem of the single-electron Landau Hamiltonian and the degeneracy of finite-volume ground states in a quantum Hall system of an interacting electrons gas. The degeneracy was found by Yoshioka, Halperin and Lee.⁽²⁸⁾ For related works, see refs. 29.

3.1. The Single-Electron Landau Hamiltonian in Two Dimensions

Consider the eigenvalue problem of the single-electron Hamiltonian

$$\mathcal{H} = \frac{1}{2m_e} [(p_x - eBy)^2 + p_y^2] \quad (3.1)$$

on the infinite plane \mathbf{R}^2 . In order to obtain an eigenvector of the Hamiltonian \mathcal{H} , put its form as

$$\phi(x, y) = e^{ikx}v(y) \quad (3.2)$$

with a wavenumber $k \in \mathbf{R}$. Substituting this into the Schrödinger equation $\mathcal{H}\phi = \mathcal{E}\phi$, one has

$$\left[\frac{1}{2m_e} (hk - eBy)^2 + \frac{1}{2m_e} p_y^2 \right] v(y) = \mathcal{E}v(y) \quad (3.3)$$

Clearly this is identical to the eigenvalue equation of a quantum harmonic oscillator as

$$\left[-\frac{\hbar^2}{2m_e} \frac{\partial^2}{\partial y^2} + \frac{e^2 B^2}{2m_e} \left(y - \frac{\hbar k}{eB} \right)^2 \right] v(y) = \mathcal{E}v(y) \quad (3.4)$$

The eigenvectors are

$$v_{n,k}(y) := v_n(y - y_k) := N_n \exp[-(y - y_k)^2/(2\ell_B^2)] H_n[(y - y_k)/\ell_B] \quad (3.5)$$

where H_n is the Hermite polynomial, $y_k = \hbar k/eB$, and N_n is the positive normalization constant so that

$$\int_{-\infty}^{+\infty} dy |v_{n,k}(y)|^2 = 1 \quad (3.6)$$

The eigenvalues are given by

$$\mathcal{E}_{n,k} = (n + \frac{1}{2}) \hbar \omega_c \quad \text{for } n = 0, 1, 2, \dots \quad (3.7)$$

with $\omega_c = eB/m_e$. Thus the eigenvectors of the Hamiltonian \mathcal{H} of (3.1) are given by

$$\phi_{n,k}(x, y) = e^{ikx} v_{n,k}(y) \quad (3.8)$$

Next we consider a single electron in $L_x \times L_y$ rectangular box $S = [-L_x/2, L_x/2] \times [-L_y/2, L_y/2]$ with $L_x L_y = 2\pi M \ell_B^2$ with an even integer M . We impose periodic boundary conditions

$$\phi(x, y) = t^{(x)}(L_x) \phi(x, y), \quad \phi(x, y) = t^{(y)}(L_y) \phi(x, y) \quad (3.9)$$

for wavefunctions ϕ on \mathbf{R}^2 . We claim that, if f satisfies (3.9), then the functions

$$f_1(x, y) = t^{(x)}(x') f(x, y) \quad (3.10)$$

and

$$f_2(x, y) = t^{(y)}(y') f(x, y) \quad (3.11)$$

also satisfy the same periodic boundary conditions. As a result, x' and y' are restricted into the following values:

$$x' = m \Delta x \quad \text{with } m \in \mathbf{Z}, \quad \text{and} \quad y' = n \Delta y \quad \text{with } n \in \mathbf{Z} \quad (3.12)$$

where

$$\Delta x := \frac{\hbar}{eB} \frac{1}{L_y}, \quad \text{and} \quad \Delta y := \frac{\hbar}{eB} \frac{1}{L_x} \quad (3.13)$$

One can easily show these statements. In fact one has

$$\begin{aligned} f_1(x, y) &= f(x - x', y) \\ &= \exp[iL_y(x - x')/\ell_B^2] f(x - x', y - L_y) \\ &= \exp[-iL_y x'/\ell_B^2] \exp[iL_y x/\ell_B^2] f(x - x', y - L_y) \\ &= \exp[-iL_y x'/\ell_B^2] \exp[iL_y x/\ell_B^2] f_1(x, y - L_y) \\ &= \exp[-iL_y x'/\ell_B^2] t^{(y)}(L_y) f_1(x, y) \\ &= \exp[-iL_y x'/\ell_B^2] f_1(x, y) \end{aligned} \quad (3.14)$$

by the definitions. This implies $L_y x' / \ell_B^2 = 2\pi m$ with an integer m . Similarly

$$\begin{aligned}
 f_2(x, y) &= \exp[iy'x/\ell_B^2] f(x, y - y') \\
 &= \exp[iy'x/\ell_B^2] f(x - L_x, y - y') \\
 &= \exp[iy'L_x/\ell_B^2] \exp[iy'(x - L_x)/\ell_B^2] f(x - L_x, y - y') \\
 &= \exp[iy'L_x/\ell_B^2] f_2(x - L_x, y) \\
 &= \exp[iy'L_x/\ell_B^2] t^{(x)}(L_x) f_2(x, y) \\
 &= \exp[iy'L_x/\ell_B^2] f_2(x, y)
 \end{aligned} \tag{3.15}$$

Thus $y'L_x/\ell_B^2 = 2\pi n$ with an integer n . Throughout the present paper we restrict the ranges of the variables x' , y' in the magnetic translations to these values of (3.12).

Since

$$t^{(y)}(y')(p_x - eBy)[t^{(y)}(y')]^{-1} = p_x - eBy \tag{3.16}$$

for any y' , the Hamiltonian \mathcal{H} of (3.1) is invariant under all the magnetic translations $t^{(x)}(\dots)$ and $t^{(y)}(\dots)$. Consider wavefunctions

$$\phi_{n,k}^P(x, y) = L_x^{-1/2} \sum_{\ell=-\infty}^{+\infty} e^{i(k+\ell K)x} v_{n,k}(y - \ell L_y) \tag{3.17}$$

for $k = 2\pi m/L_x$ with $m = -M/2 + 1, \dots, M/2 - 1, M/2$, and with $K = L_y/\ell_B^2$. These wavefunctions are eigenvectors of the Hamiltonian \mathcal{H} of (3.1) satisfying the periodic boundary conditions (3.9), because $L_x L_y = 2\pi M \ell_B^2$ with the even integer M . The eigenvalues of $\phi_{n,k}^P$ are given by (3.7). We identify the integer m of a wavenumber k with a lattice site m in the one-dimensional lattice $\{-M/2 + 1, -M/2 + 2, \dots, M/2 - 1, M/2\}$, with the periodic boundary conditions. Then there are $(n_{\max} + 1)$ atomic levels at each site in the present quantum Hall system because we have restricted the Hilbert space to the lowest $(n_{\max} + 1)$ Landau levels. An observable at a site in the system can be expressed by a $(n_{\max} + 2) \times (n_{\max} + 2)$ matrix. Therefore the present quantum Hall system is equivalent to a one-dimensional spinless fermion system with long-range interactions. Here we should remark that the lattice constant is given by $\Delta y = 2\pi\hbar/(eBL_x)$ which tends to zero as $L_x \rightarrow \infty$. This causes us a technical problem for taking the limit $L_x \rightarrow \infty$ as we will show in Section 6. This is why we must fix L_x to a finite value.

In the rest of the present section, we review the properties of the eigenfunctions (3.17) and check the completeness of the system of the eigenfunctions.

One can easily get the following lemma:

Lemma 3.1. The vector $\phi_{n,k}^P$ of (3.17) is an eigenvector of the magnetic translation $t^{(x)}(\Delta x)$, i.e.,

$$t^{(x)}(\Delta x) \phi_{n,k}^P = e^{-ik\Delta x} \phi_{n,k}^P = e^{-i2\pi m/M} \phi_{n,k}^P \quad \text{with} \quad k = \frac{2\pi m}{L_x} \quad (3.18)$$

and the magnetic translation $t^{(y)}(\Delta y)$ shifts the wavenumber k of the vector $\phi_{n,k}^P$ by one unit $2\pi/L_x$ as

$$t^{(y)}(\Delta y) \phi_{n,k}^P = \phi_{n,k'}^P \quad \text{with} \quad k' = k + \frac{\Delta y}{\ell_B^2} = k + \frac{2\pi}{L_x} \quad (3.19)$$

As usual we denote by $L^2(S)$ the set of functions f on S such that

$$\int_S dx dy |f(x, y)|^2 = \int_{-L_x/2}^{L_x/2} dx \int_{-L_y/2}^{L_y/2} dy |f(x, y)|^2 < \infty \quad (3.20)$$

Further we define the associate inner product (f, g) as

$$(f, g) = \int_S dx dy [f(x, y)]^* g(x, y) \quad (3.21)$$

for $f, g \in L^2(S)$.

Lemma 3.2. Let f, g be functions on \mathbf{R}^2 such that $f, g \in L^2(S)$, and that f, g satisfy the boundary conditions (3.9). Then

$$(f, g) = \int_{-L_x/2}^{L_x/2} dx \int_{-L_y/2+y_0}^{L_y/2+y_0} dy [f(x, y)]^* g(x, y) \quad (3.22)$$

for any $y_0 \in \mathbf{R}$.

Proof. By the periodic boundary condition $f(x, y) = t^{(x)}(L_x) f(x, y)$, the function f can be expanded in Fourier series as

$$f(x, y) = L_x^{-1/2} \sum_k e^{ikx} \hat{f}(k, y) \quad (3.23)$$

Further, since

$$\begin{aligned} f(x, y) &= t^{(y)}(L_y) f(x, y) = L_x^{-1/2} \sum_k e^{i(k+K)x} \hat{f}(k, y - L_y) \\ &= L_x^{-1/2} \sum_k e^{ikx} \hat{f}(k - K, y - L_y) \end{aligned} \quad (3.24)$$

the following relation holds:

$$\hat{f}(k, y) = \hat{f}(k - K, y - L_y) \quad (3.25)$$

Using this relation repeatedly, the function f of (3.23) can be rewritten as

$$f(x, y) = \sum_{\{k = 2\pi n/L_x \mid -M/2 + 1 \leq n \leq M/2\}} L_x^{-1/2} \sum_{\ell = -\infty}^{+\infty} e^{i(k + \ell K)x} \hat{f}(k, y - \ell L_y) \quad (3.26)$$

This expression yields

$$\begin{aligned} (f, g) &= \int_{-L_x/2}^{L_x/2} dx \int_{-L_y/2}^{L_y/2} dy [f(x, y)]^* g(x, y) \\ &= \sum_{\{k = 2\pi n/L_x \mid -M/2 + 1 \leq n \leq M/2\}} \sum_{\ell = -\infty}^{+\infty} \int_{-L_y/2}^{L_y/2} dy [\hat{f}(k, y - \ell L_y)]^* \\ &\quad \times \hat{g}(k, y - \ell L_y) \\ &= \sum_{\{k = 2\pi n/L_x \mid -M/2 + 1 \leq n \leq M/2\}} \int_{-\infty}^{+\infty} dy [\hat{f}(k, y)]^* \hat{g}(k, y) \\ &= \sum_{\{k = 2\pi n/L_x \mid -M/2 + 1 \leq n \leq M/2\}} \sum_{\ell = -\infty}^{+\infty} \int_{-L_y/2 + y_0}^{L_y/2 + y_0} dy \\ &\quad \times [\hat{f}(k, y - \ell L_y)]^* \hat{g}(k, y - \ell L_y) \\ &= \int_{-L_x/2}^{L_x/2} dx \int_{-L_y/2 + y_0}^{L_y/2 + y_0} dy [f(x, y)]^* g(x, y) \quad \blacksquare \end{aligned} \quad (3.27)$$

Let us check that the set of the eigenvectors $\{\phi_{n,k}^P\}$ of (3.17) forms an orthonormal complete system. From the third equality in (3.27) in the proof of Lemma 3.2, the orthogonality is valid as

$$(\phi_{n',k'}^P, \phi_{n,k}^P) = \int_{-\infty}^{+\infty} dy v_{n',k'}^*(y) v_{n,k}(y) \delta_{k,k'} = \delta_{n,n'} \delta_{k,k'} \quad (3.28)$$

Here $\delta_{k,k'}$ is the Kronecker delta. To show the completeness, consider a function f satisfying the boundary conditions (3.9). In the same way,

$$(\phi_{n,k}^P, f) = \int_{-\infty}^{+\infty} dy v_{n,k}^*(y) \hat{f}(k, y) \quad (3.29)$$

This implies that the function f must be zero if the inner product $(\phi_{n,k}^P, f)$ is vanishing for all the vectors $\phi_{n,k}^P$.

3.2. Degeneracy of Finite-Volume Ground States

In this section, we review the degeneracy⁽²⁸⁾ of the finite-volume ground states of a quantum Hall system. A wide class of quantum Hall systems without disorder has the property. As an example, we consider an interacting N electrons gas in a uniform magnetic field, whose Hamiltonian is given by

$$H^{(N)} = \sum_{j=1}^N \frac{1}{2m_e} [(p_{x,j} - eBy_j)^2 + p_{y,j}^2] + U^{(N)}(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) \quad (3.30)$$

which is the Hamiltonian $H^{(N)}$ of (2.1) with no single-body potential W . Clearly the system has the translational invariance.

To begin with, we recall the properties of the magnetic translations. From the definitions (2.12) of the magnetic translations $t^{(x)}(\dots)$ and $t^{(y)}(\dots)$, one can easily get

$$\begin{aligned} t^{(x)}(x') t^{(y)}(y') f(x, y) &= t^{(x)}(x') \exp[iy'x/\ell_B^2] f(x, y - y') \\ &= \exp[iy'(x - x')/\ell_B^2] f(x - x', y - y') \\ &= \exp[-ix'y'/\ell_B^2] t^{(y)}(y') t^{(x)}(x') f(x, y) \end{aligned} \quad (3.31)$$

for a function f . This implies

$$t^{(x)}(x') t^{(y)}(y') = \exp[-ix'y'/\ell_B^2] t^{(y)}(y') t^{(x)}(x') \quad (3.32)$$

We define the magnetic translations $T^{(N, x)}(x')$ and $T^{(N, y)}(y')$ for an N electrons state as

$$T^{(N, x)}(x') = \bigotimes_{j=1}^N t_j^{(x)}(x') \quad (3.33)$$

and

$$T^{(N, y)}(y') = \bigotimes_{j=1}^N t_j^{(y)}(y') \quad (3.34)$$

From the commutation relation (3.32), one has

$$T^{(N, x)}(x') T^{(N, y)}(y') = \exp[-ix'y'N/\ell_B^2] T^{(N, y)}(y') T^{(N, x)}(x') \quad (3.35)$$

In particular,

$$T^{(N, x)}(\Delta x) T^{(N, y)}(\Delta y) = \exp[-i2\pi\nu] T^{(N, y)}(\Delta y) T^{(N, x)}(\Delta x) \quad (3.36)$$

where $\nu = N/M$ with $M = L_x L_y eB/h$. The number ν is nothing but the filling factor for the Landau levels.

Note that all the magnetic translations $T^{(N,y)}(\dots)$ and $T^{(N)}(\dots)$ commute with the Hamiltonian $H^{(N)}$ of (3.30). Let $\Phi^{(N)}$ be a simultaneous eigenvector of the Hamiltonian $H^{(N)}$ and the magnetic translation operator $T^{(N,y)}(\Delta y)$, i.e.,

$$H^{(N)}\Phi^{(N)} = E^{(N)}\Phi^{(N)}, \quad T^{(N,y)}(\Delta y)\Phi^{(N)} = e^{i2\pi n/M}\Phi^{(N)}, \quad \text{with } n \in \mathbf{Z} \quad (3.37)$$

where $E^{(N)}$ is the energy eigenvalue. Let $\Psi^{(N)} = T^{(N,x)}(\Delta x)\Phi^{(N)}$. Then the vector $\Psi^{(N)}$ is an eigenvector of $H^{(N)}$ with the same eigenvalue $E^{(N)}$. Further one can easily show

$$\begin{aligned} T^{(N,y)}(\Delta y)\Psi^{(N)} &= T^{(N,y)}(\Delta y)T^{(N,x)}(\Delta x)\Phi^{(N)} \\ &= e^{i2\pi\nu}T^{(N,x)}(\Delta x)T^{(N,y)}(\Delta y)\Phi^{(N)} \\ &= e^{i2\pi\nu}e^{i2\pi n/M}T^{(N,x)}(\Delta x)\Phi^{(N)} \\ &= e^{i2\pi\nu}e^{i2\pi n/M}\Psi^{(N)} \end{aligned} \quad (3.38)$$

by using the commutation relation (3.36). Thus $\Psi^{(N)}$ is also an eigenvector of $T^{(N,y)}(\Delta y)$. From these observations, one can notice the fact that, if $\nu = p/q$ with mutually prime positive integers p and q , then any energy level of finite volume is at least q -fold degenerate.

4. THE LIEB-SCHULTZ-MATTIS METHOD

In this section, we construct a candidate for a low energy excitation above a ground state by using the Lieb-Schultz-Mattis method.⁽⁶⁾ Our goal is to give the proofs of our main Theorems 2.3, 2.4 and 2.5. For the convenience of readers, technical estimates in the proofs are given in Sections 5 and 6 and Appendices B, C, D and E.

We denote by $\mathbf{H}_{L_y}^{(N)}(n_{\max})$ the restricted N electrons Hilbert space to the lowest $(n_{\max} + 1)$ Landau levels with the system size L_y in the y direction. Throughout this section, we fix n_{\max} and L_x (the system size in the x direction) to large numbers.

Let $\Phi_{L_y}^{(N)}$ be a normalized N electrons vector in $\mathbf{H}_{L_y}^{(N)}(n_{\max})$. We expand $\Phi_{L_y}^{(N)}$ as

$$\Phi_{L_y}^{(N)} = \sum_{\{\xi_j\}} a(\{\xi_j\}) \text{Asym}[\phi_{\xi_1}^P \otimes \phi_{\xi_2}^P \otimes \cdots \otimes \phi_{\xi_N}^P] \quad (4.1)$$

in terms of the eigenvectors $\phi_{n,k}^P$ of (3.17) for the single-electron Hamiltonian \mathcal{H} of (3.1). Here we have written

$$\xi_j = (n_j, k_j) = (n_j, 2\pi m_j/L_x) \quad \text{for } j = 1, 2, \dots, N \quad (4.2)$$

i.e., $\phi_{\xi_j}^P = \phi_{n_j, k_j}^P$, and $\text{Asym}[\dots]$ stands for the antisymmetrization. Note that we have

$$T^{(N,x)}(\Delta x) \Phi_{L_y}^{(N)} = \sum_{\{\xi_j\}} a(\{\xi_j\}) \left[\prod_{j=1}^N e^{-i2\pi m_j/M} \right] \text{Asym}[\phi_{\xi_1}^P \otimes \phi_{\xi_2}^P \otimes \dots \otimes \phi_{\xi_N}^P] \quad (4.3)$$

from Lemma 3.1. This vector $T^{(N,x)}(\Delta x) \Phi_{L_y}^{(N)}$ is a vector globally twisting the quantum mechanical phase for $\Phi_{L_y}^{(N)}$. As we saw in the preceding section, if $\Phi_{L_y}^{(N)}$ is a ground state of the Hamiltonian $H^{(N)}$ of (3.30), the vector $T^{(N,x)}(\Delta x) \Phi_{L_y}^{(N)}$ is a ground state, too. As Haldane pointed out,⁽²³⁾ the degeneracy of the ground states does not directly lead to physical significance because the degeneracy is related to the degree of freedom for the center of the total mass. But we can construct a physically natural low energy excitation above a ground state for the present Hamiltonian $H^{(N)}(n_{\max})$ of (2.18), by combining the translational invariance in the y direction with the Lieb–Schultz–Mattis method. To do this, we replace the globally twisting phase change of (4.3) with a local one. Namely we construct a locally perturbed state for a state $\Phi_{L_y}^{(N)}$ which is not necessarily a ground state.

For this purpose, we introduce a unitary transformation $U_{\pm, q}^{(\ell)}$ with a compact support for $\ell, q \in \mathbf{N}$ as

$$U_{\pm, q}^{(\ell)} \Phi_{L_y}^{(N)} := \sum_{\{\xi_j\}} a(\{\xi_j\}) \exp \left[\pm i2\pi \sum_{j=1}^N \tilde{m}(m_j)/\ell \right] \times \text{Asym}[\phi_{\xi_1} \otimes \phi_{\xi_2} \otimes \dots \otimes \phi_{\xi_N}] \quad (4.4)$$

where

$$\tilde{m}(m) := \begin{cases} n, & \text{if } (n-1)q < m \leq nq \\ 0, & \text{otherwise} \end{cases} \quad \text{with } n = 1, 2, \dots, \ell \quad (4.5)$$

for $m \in \mathbf{Z}$. Consider two vectors

$$\Psi_{\pm, L_y}^{(N)} := U_{\pm, q}^{(\ell)} \Phi_{L_y}^{(N)} \quad (4.6)$$

which are locally perturbed vectors for $\Phi_{L_y}^{(N)}$ of (4.1), and

$$\Delta E_{L_y}^{(N)} = \eta_{L_y}^{(N)}(H_{L_y}^{(N)}(n_{\max})) - \omega_{L_y}^{(N)}(H_{L_y}^{(N)}(n_{\max})) \quad (4.7)$$

where

$$\eta_{L_y}^{(N)}(\dots) = \frac{1}{2} \langle \Psi_{+, L_y}^{(N)}, (\dots) \Psi_{+, L_y}^{(N)} \rangle + \frac{1}{2} \langle \Psi_{-, L_y}^{(N)}, (\dots) \Psi_{-, L_y}^{(N)} \rangle \quad (4.8)$$

and

$$\omega_{L_y}^{(N)}(\dots) = \langle \Phi_{L_y}^{(N)}, (\dots) \Phi_{L_y}^{(N)} \rangle \quad (4.9)$$

Here $H_{L_y}^{(N)}(n_{\max})$ is the Hamiltonian $H^{(N)}(n_{\max})$ of (2.18) with the system size L_y in the y direction. The vectors $\Psi_{\pm, L_y}^{(N)}$ are candidates for natural low energy excitations when $\Phi_{L_y}^{(N)}$ leads to an infinite-volume ground state. When $\Phi_{L_y}^{(N)}$ is a finite-volume ground state with the energy eigenvalue $E_{L_y}^{(N)}$, we have

$$\begin{aligned} \Delta E_{L_y}^{(N)} &= \frac{1}{2} \langle \Psi_{+, L_y}^{(N)}, (H_{L_y}^{(N)}(n_{\max}) - E_{L_y}^{(N)}) \Psi_{+, L_y}^{(N)} \rangle \\ &\quad + \frac{1}{2} \langle \Psi_{-, L_y}^{(N)}, (H_{L_y}^{(N)}(n_{\max}) - E_{L_y}^{(N)}) \Psi_{-, L_y}^{(N)} \rangle \end{aligned} \quad (4.10)$$

Thus $\Delta E_{L_y}^{(N)}$ gives an upper bound for the energy gap.

Lemma 4.1. For any given small $\varepsilon > 0$, there exist ℓ and L such that

$$|\Delta E_{L_y}^{(N)}| \leq \varepsilon \quad \text{for any } L_y \geq L \quad (4.11)$$

This lemma gives an estimate for an energy gap above a ground state. The proof is given in Section 6.

Next we study a condition for which $\Psi_{\pm, L_y}^{(N)}$ is orthogonal to $\Phi_{L_y}^{(N)}$. We define a local charge operator $\hat{n}_{m, n}$ as

$$\hat{n}_{n, m} \phi_{n', k'}^{\mathbf{P}} = \delta_{n, n'} \delta_{m, m'} \phi_{n', k'}^{\mathbf{P}} \quad (4.12)$$

with $k' = 2\pi m' / L_x$. Further we define

$$\hat{n}_m := \sum_{n=0}^{n_{\max}} \hat{n}_{n, m} \quad (4.13)$$

Proposition 4.2. Let $\Phi_{L_y}^{(N)} \in \mathbf{H}_{L_y}^{(N)}(n_{\max})$ be a normalized eigenvector of the magnetic translation $T^{(N, y)}(q \Delta y)$ with $q \in \mathbf{N}$. Write

$$\omega(\dots) = w^* - \lim_{L_y \rightarrow \infty} \langle \Phi_{L_y}^{(N)}, (\dots) \Phi_{L_y}^{(N)} \rangle \quad (4.14)$$

where the weak limit $L_y \rightarrow \infty$ is taken for a fixed L_x and a fixed filling factor ν . Suppose that the infinite-volume state ω satisfies

$$\lim_{\ell \rightarrow \infty} \frac{1}{\ell^2} \sum_{i, j=1}^{\ell} \omega(\hat{n}_i \hat{n}_j) = \nu^2 \quad (4.15)$$

Then

$$\lim_{\ell \rightarrow \infty} \omega(U_{\pm, q}^{(\ell)}) = 0 \quad (4.16)$$

for $qv \notin \mathbf{N}$.

The proof is given in Section 5. The idea of the proof is due to Hal Tasaki.⁽³⁰⁾ From the proof, one can see that the statement of Proposition 4.2 holds for a wide class of systems with translational invariance.

Before giving the proofs of our main Theorems 2.3, 2.4 and 2.5, we recall the GNS representation of a C^* algebra \mathcal{A} on a Hilbert space.¹⁶ Let ω be an infinite-volume state. Then there exist a Hilbert space \mathbf{H}_ω , a normalized vector Ω_ω and a representation π_ω of \mathcal{A} on \mathbf{H}_ω such that

$$\omega(a) = (\Omega_\omega, \pi_\omega(a) \Omega_\omega) \quad \text{for any } a \in \mathcal{A} \quad (4.17)$$

Here, if ω is a ground state, there exist a self-adjoint operator $H_\omega \geq 0$ on \mathbf{H}_ω such that

$$H_\omega \Omega_\omega = 0, \quad e^{itH_\omega/\hbar} \pi_\omega(a) e^{-itH_\omega/\hbar} = \pi_\omega(\tau_t(a)) \quad \text{for any } a \in \mathcal{A} \quad (4.18)$$

Namely H_ω is the Hamiltonian in the infinite volume limit. Conversely, if the vector Ω_ω satisfies the conditions (4.18) for a self-adjoint operator $H_\omega \geq 0$ on \mathbf{H}_ω , then the corresponding state $\omega(\dots) = (\Omega_\omega, \pi_\omega(\dots) \Omega_\omega)$ is a ground state. Using this representation, the gapful condition (2.43) in Definition 2.2 can be written as

$$(\Omega_\omega, [\pi_\omega(a)]^* [\hat{f}(H_\omega/\hbar)]^2 \pi_\omega(a) \Omega_\omega) = 0 \quad \text{for any } a \in \mathcal{A} \quad (4.19)$$

Proof of Theorem 2.3. Let $\Phi_{L_y}^{(N)}$ be a normalized ground state of the Hamiltonian $H_{L_y}^{(N)}(n_{\max})$ of (2.18) and eigenvector of $T^{(N, \nu)}(\Delta y)$, i.e., a translationally invariant ground state for a finite volume. We fix the filling factor ν to a non-integer. Let $\tilde{\Phi}_A^{(N)}$ be the corresponding vector in the Fock space $\mathbf{H}_{L_y}(n_{\max}) := \bigoplus_{N \geq 0} \mathbf{H}_{L_y}^{(N)}(n_{\max})$. We denote by ω the infinite-volume ground state, i.e.,

$$\omega(\dots) = w^* - \lim_{A \uparrow \mathbf{Z}} \langle \tilde{\Phi}_A^{(N)}, (\dots) \tilde{\Phi}_A^{(N)} \rangle \quad (4.20)$$

¹⁶ For the GNS construction of a representation of a C^* algebra, see ref. 22.

for $\mathcal{A}_{\text{loc}}^{U(1)}$. By Theorem 2.1, there exists a chemical potential μ such that ω is an infinite-volume ground state for \mathcal{A}_{loc} . Assume that ω is the unique ground state with the chemical potential μ . Since a unique pure ground state has the clustering property⁽²²⁾

$$\omega(\hat{n}_i \hat{n}_j) - \omega(\hat{n}_i) \omega(\hat{n}_j) \rightarrow 0 \quad \text{as } |i-j| \rightarrow \infty \quad (4.21)$$

we have

$$\begin{aligned} 0 &= \lim_{\ell \rightarrow \infty} \lim_{L_y \rightarrow \infty} \langle \Phi_{L_y}^{(N)}, U_{\pm,1}^{(\ell)} \Phi_{L_y}^{(N)} \rangle \\ &= \lim_{\ell \rightarrow \infty} \omega(\tilde{U}_{\pm,1}^{(\ell)}) \\ &= \lim_{\ell \rightarrow \infty} (\Omega_\omega, \pi_\omega(\tilde{U}_{\pm,1}^{(\ell)}) \Omega_\omega) \end{aligned} \quad (4.22)$$

from Proposition 4.2. Here $\tilde{U}_{\pm,1}^{(\ell)}$ is the extension of $U_{\pm,1}^{(\ell)}$ to that in the Fock space, π_ω is the GNS representation of \mathcal{A} on the Hilbert space \mathbf{H}_ω , and $\Omega_\omega \in \mathbf{H}_\omega$ is the ground state corresponding to ω . This implies that the vectors $[\pi_\omega(\tilde{U}_{\pm,1}^{(\ell)}) - (\Omega_\omega, \pi_\omega(\tilde{U}_{\pm,1}^{(\ell)}) \Omega_\omega)] \Omega_\omega$ are excitations above the unique ground state Ω_ω for a large ℓ . Clearly the norms of these vectors go to one as $\ell \rightarrow \infty$.

Next we show the existence of a gapless excitation. Note that

$$\begin{aligned} &\langle \Phi_{L_y}^{(N)}, (U_{\pm,1}^{(\ell)})^* H_{L_y}^{(N)}(n_{\max}) U_{\pm,1}^{(\ell)} \Phi_{L_y}^{(N)} \rangle - \langle \Phi_{L_y}^{(N)}, H_{L_y}^{(N)}(n_{\max}) \Phi_{L_y}^{(N)} \rangle \\ &= \langle \Phi_{L_y}^{(N)}, (U_{\pm,1}^{(\ell)})^* [H_{L_y}^{(N)}(n_{\max}), U_{\pm,1}^{(\ell)}] \Phi_{L_y}^{(N)} \rangle \\ &= \langle \tilde{\Phi}_A^{(N)}, (\tilde{U}_{\pm,1}^{(\ell)})^* [\tilde{H}_A(n_{\max}), \tilde{U}_{\pm,1}^{(\ell)}] \tilde{\Phi}_A^{(N)} \rangle \\ &= \langle \tilde{\Phi}_A^{(N)}, (\tilde{U}_{\pm,1}^{(\ell)})^* [\tilde{H}_{A,\mu}(n_{\max}), \tilde{U}_{\pm,1}^{(\ell)}] \tilde{\Phi}_A^{(N)} \rangle \end{aligned} \quad (4.23)$$

Further we have

$$\begin{aligned} &\lim_{A \uparrow \mathbf{Z}} \omega((\tilde{U}_{\pm,1}^{(\ell)})^* [\tilde{H}_{A,\mu}(n_{\max}), \tilde{U}_{\pm,1}^{(\ell)}]) \\ &= (\Omega_\omega, [\pi_\omega(\tilde{U}_{\pm,1}^{(\ell)})]^* H_\omega \pi_\omega(\tilde{U}_{\pm,1}^{(\ell)}) \Omega_\omega) \end{aligned} \quad (4.24)$$

in the thermodynamic limit because

$$\lim_{A \uparrow \mathbf{Z}} \omega(a^* [\tilde{H}_{A,\mu}(n_{\max}), a]) = (\Omega_\omega, [\pi_\omega(a)]^* [H_\omega, \pi_\omega(a)] \Omega_\omega) \quad (4.25)$$

for any observable a in a domain for the commutator.¹⁷ Combining these observations with Lemma 4.1, we have the following: For any given small $\varepsilon > 0$, there exists ℓ such that

$$(\Omega_\omega, [\pi_\omega(\tilde{U}_{\pm,1}^{(\ell)})]^* H_\omega \pi_\omega(\tilde{U}_{\pm,1}^{(\ell)}) \Omega_\omega) \leq \varepsilon \quad (4.26)$$

This implies that there exists a gapless excitation above the unique ground state. ■

Proof of Theorem 2.4. Let the filling factor ν be a non-integer, and let ω be a pure ground state with a non-zero excitation gap. Assume that all the infinite-volume ground states are translationally invariant with the period of one lattice unit, and we will find a contradiction. Without loss of generality, we can assume that there exists a sequence of vectors $\{\tilde{\Phi}_A\}$ such that

$$\omega(\dots) = w^* - \lim_{A \uparrow \mathbf{Z}} \langle \tilde{\Phi}_A, (\dots) \tilde{\Phi}_A \rangle \quad (4.27)$$

Each vector $\tilde{\Phi}_A$ for a finite lattice A is expanded as

$$\tilde{\Phi}_A = \sum_N \alpha_N \tilde{\Phi}_A^{(N)} \quad (4.28)$$

in terms of the N electrons vectors $\tilde{\Phi}_A^{(N)}$. Then we can assume, by the assumption about the translational invariance, that the expectation $\langle \tilde{\Phi}_A^{(N)}, (\dots) \tilde{\Phi}_A^{(N)} \rangle$ is translationally invariant with the period 1. Using the expansion, we have

$$\sin^2(\pi\nu) |\omega(\tilde{U}_{\pm,1}^{(\ell)})|^2 \leq \pi^2 \left[\frac{1}{\ell^2} \sum_{s,t=1}^{\ell} \omega(\tilde{n}_s \tilde{n}_t) - \nu^2 \right] \quad (4.29)$$

in the same way as in the proof of Proposition 4.2. Here \tilde{n}_j is the number operator corresponding to \hat{n}_j . Since the ground state ω has the clustering property due to the purity, we obtain

$$0 = \lim_{\ell \rightarrow \infty} \omega(\tilde{U}_{\pm,1}^{(\ell)}) = \lim_{\ell \rightarrow \infty} (\Omega_\omega, \pi_\omega(\tilde{U}_{\pm,1}^{(\ell)}) \Omega_\omega) \quad (4.30)$$

from (4.29) with the assumption $\nu \notin \mathbf{N}$. Here π_ω is the GNS representation of \mathcal{A} on the Hilbert space \mathbf{H}_ω , and Ω_ω is the ground state corresponding to the state ω .

¹⁷ Roughly speaking, the Hamiltonian H_ω in the infinite volume limit is defined by the relation (4.25) because the left-hand side of (4.25) is non-negative for any $a \in \mathcal{A}_{\text{loc}}$. See ref. 22 for the details.

Consider a vector $\Xi = (1 - G) \pi_\omega(\tilde{U}_{\pm,1}^{(\ell)}) \Omega_\omega$, where G is the orthogonal projection onto the sector of the ground states. We want to show that the norm of Ξ is non-vanishing in the limit $\ell \rightarrow \infty$. Assume this is not true, and we find a contradiction. This assumption is rephrased as follows: For any given small $\varepsilon > 0$, there exist a positive integer ℓ_0 such that

$$|(\Omega_{\omega'}, \pi_\omega(\tilde{U}_{\pm,1}^{(\ell)}) \Omega_\omega) - 1| < \varepsilon \quad \text{for any } \ell > \ell_0 \quad (4.31)$$

where $\Omega_{\omega'} \in \mathbf{H}_\omega$ is a normalized ground state which may depend on the integer ℓ . We decompose $\Omega_{\omega'}$ as

$$\Omega_{\omega'} = c \pi_\omega(\tilde{U}_{\pm,1}^{(\ell)}) \Omega_\omega + \Omega' \quad \text{with } (\Omega', \pi_\omega(\tilde{U}_{\pm,1}^{(\ell)}) \Omega_\omega) = 0 \quad (4.32)$$

where c is a complex number. Immediately, we have

$$|1 - c| < \varepsilon, \quad \|\Omega'\| \leq \sqrt{2\varepsilon} \quad (4.33)$$

Using these inequalities, we get

$$\begin{aligned} & \|\omega'(\dots) - \omega([\tilde{U}_{\pm,1}^{(\ell)}]^*(\dots) \tilde{U}_{\pm,1}^{(\ell)})\| \\ &= \|(\Omega_{\omega'}, (\dots) \Omega_{\omega'}) - (\Omega_\omega, [\pi_\omega(\tilde{U}_{\pm,1}^{(\ell)})]^*(\dots) \pi_\omega(\tilde{U}_{\pm,1}^{(\ell)}) \Omega_\omega)\| \leq \varepsilon' \end{aligned} \quad (4.34)$$

where $\varepsilon' = 2(2\varepsilon + \sqrt{2\varepsilon})$. Since ω' and ω are translationally invariant by the assumption, we have

$$\begin{aligned} \|a\| \varepsilon' &\geq |\omega'(\tau_{-j}^{(y)}(a)) - \omega([\tilde{U}_{\pm,1}^{(\ell)}]^* \tau_{-j}^{(y)}(a) \tilde{U}_{\pm,1}^{(\ell)})| \\ &= |\omega'(a) - \omega(\tau_j^{(y)}([\tilde{U}_{\pm,1}^{(\ell)}]^* \tau_{-j}^{(y)}(a) \tilde{U}_{\pm,1}^{(\ell)}))| \\ &= |\omega'(a) - \omega(\tau_j^{(y)}([\tilde{U}_{\pm,1}^{(\ell)}]^*) a \tau_j^{(y)}(\tilde{U}_{\pm,1}^{(\ell)}))| \end{aligned} \quad (4.35)$$

for any $a \in \mathcal{A}_{\text{loc}}$. In the limit $j \rightarrow \infty$, we get

$$|\omega'(a) - \omega(a)| = |(\Omega_{\omega'}, \pi_\omega(a) \Omega_{\omega'}) - (\Omega_\omega, \pi_\omega(a) \Omega_\omega)| \leq \varepsilon' \|a\| \quad (4.36)$$

for any $a \in \mathcal{A}_{\text{loc}}$. We decompose $\Omega_{\omega'}$ as

$$\Omega_{\omega'} = d \Omega_\omega + \Omega'' \quad \text{with } (\Omega'', \Omega_\omega) = 0 \quad (4.37)$$

where d is a complex number. Taking the orthogonal projection onto Ω_ω and the orthogonal projection onto Ω'' as observables for¹⁸ (4.36), we obtain

$$1 - |d|^2 \leq \varepsilon', \quad \|\Omega''\|^2 \leq \varepsilon' \quad (4.38)$$

Substituting these inequalities and the decomposition (4.37) into (4.31), we have

$$|d^*(\Omega_\omega, \pi_\omega(\tilde{U}_{\pm,1}^{(\ell)}) \Omega_\omega) - 1| \leq \varepsilon + \sqrt{\varepsilon'} \quad (4.39)$$

This inequality contradicts (4.30). Thus the norm of Ξ is non-vanishing in the limit $\ell \rightarrow \infty$.

Next we show that the vector Ξ gives a low energy excitation. Note that

$$\begin{aligned} & \langle \tilde{\Phi}_A, [\tilde{U}_{\pm,1}^{(\ell)}]^* [\tilde{H}_{A,\mu}(n_{\max}), \tilde{U}_{\pm,1}^{(\ell)}] \tilde{\Phi}_A \rangle \\ &= \sum_N |\alpha_N|^2 \langle \tilde{\Phi}_A^{(N)}, [\tilde{U}_{\pm,1}^{(\ell)}]^* [\tilde{H}_{A,\mu}(n_{\max}), \tilde{U}_{\pm,1}^{(\ell)}] \tilde{\Phi}_A^{(N)} \rangle \\ &= \sum_N |\alpha_N|^2 \langle \tilde{\Phi}_A^{(N)}, [\tilde{U}_{\pm,1}^{(\ell)}]^* [\tilde{H}_A(n_{\max}), \tilde{U}_{\pm,1}^{(\ell)}] \tilde{\Phi}_A^{(N)} \rangle \\ &= \sum_N |\alpha_N|^2 [\langle \tilde{\Phi}_A^{(N)}, [\tilde{U}_{\pm,1}^{(\ell)}]^* \tilde{H}_A(n_{\max}) \tilde{U}_{\pm,1}^{(\ell)} \tilde{\Phi}_A^{(N)} \rangle \\ & \quad - \langle \tilde{\Phi}_A^{(N)}, \tilde{H}_A(n_{\max}) \tilde{\Phi}_A^{(N)} \rangle] \end{aligned} \quad (4.40)$$

Combining this with the definition (4.7) of $\Delta E_{L_y}^{(N)}$, we have

$$\begin{aligned} & \frac{1}{2} \langle \tilde{\Phi}_A, [\tilde{U}_{+,1}^{(\ell)}]^* [\tilde{H}_{A,\mu}(n_{\max}), \tilde{U}_{+,1}^{(\ell)}] \tilde{\Phi}_A \rangle \\ & \quad + \frac{1}{2} \langle \tilde{\Phi}_A, [\tilde{U}_{-,1}^{(\ell)}]^* [\tilde{H}_{A,\mu}(n_{\max}), \tilde{U}_{-,1}^{(\ell)}] \tilde{\Phi}_A \rangle \\ &= \sum_N |\alpha_N|^2 \Delta E_{L_y}^{(N)} \end{aligned} \quad (4.41)$$

Further, by using Lemma 4.1 we obtain the following: For any given small $\varepsilon > 0$, there exists ℓ such that

$$\begin{aligned} \varepsilon & \geq \lim_{A \uparrow \mathbf{Z}} \omega([\tilde{U}_{\pm,1}^{(\ell)}]^* [\tilde{H}_{A,\mu}(n_{\max}), \tilde{U}_{\pm,1}^{(\ell)}]) \\ &= (\Omega_\omega, [\pi_\omega(\tilde{U}_{\pm,1}^{(\ell)})]^* H_\omega \pi_\omega(\tilde{U}_{\pm,1}^{(\ell)}) \Omega_\omega) \end{aligned} \quad (4.42)$$

¹⁸ Since the state ω' is extended to that for the set of all bounded operators on \mathbf{H}_ω , the inequality (4.36) is valid also for the set of all bounded operators by the Hahn–Banach theorem.⁽²²⁾

This implies the existence of a gapless excitation above the ground state ω , with the above result about the vector Ξ . Since there is no gapless excitation above ω , the assumption that all the infinite-volume ground states are translationally invariant with the period 1, is not valid. Namely a translational symmetry breaking occurs. ■

Proof of Theorem 2.5. Since the proof is very similar to that of Theorem 2.4, we roughly sketch it.

Let ω be a translationally invariant pure ground state with a period $q \in \mathbf{N}$ and with a non-zero excitation gap. Assuming $qv \notin \mathbf{N}$, we find a contradiction. In the same way as in the proof of Theorem 2.4, we have

$$0 = \lim_{\ell \rightarrow \infty} \omega(\tilde{U}_{\pm, q}^{(\ell)}) = \lim_{\ell \rightarrow \infty} (\Omega_{\omega}, \pi(\tilde{U}_{\pm, q}^{(\ell)}) \Omega_{\omega}) \quad (4.43)$$

Let $\Xi = (1 - G) \pi_{\omega}(\tilde{U}_{\pm, q}^{(\ell)}) \Omega_{\omega}$. Then the norm of the vector Ξ is non-vanishing in the limit $\ell \rightarrow \infty$ again. Further we have

$$(\Omega_{\omega}, [\pi_{\omega}(\tilde{U}_{\pm, q}^{(\ell)})]^* H_{\omega} \pi_{\omega}(\tilde{U}_{\pm, q}^{(\ell)}) \Omega_{\omega}) \leq \varepsilon \quad (4.44)$$

for large ℓ . Thus there exists a gapless excitation above the ground state ω . Since ω has a gap, the assumption $qv \notin \mathbf{N}$ is not valid. Namely $qv \in \mathbf{N}$. ■

5. ORTHOGONALITY—PROOF OF PROPOSITION 4.2

In order to prove Proposition 4.2, we first study the properties of the vectors $\Phi_{L_y}^{(N)}$ and $\Psi_{\pm, L_y}^{(N)} = U_{\pm, q}^{(\ell)} \Phi_{L_y}^{(N)}$ for the action of $T^{(N, y)}(q \Delta y)$. Note that

$$\begin{aligned} T^{(N, y)}(q \Delta y) \Phi_{L_y}^{(N)} &= \sum_{\{\xi_j\}} a(\{\xi_j\}) T^{(N, y)}(q \Delta y) \text{Asym}[\phi_{\xi_1}^P \otimes \cdots \otimes \phi_{\xi_N}^P] \\ &= \sum_{\{\xi_j\}} a(\{\xi_j\}) \text{Asym}[\phi_{\xi_1}^P \otimes \cdots \otimes \phi_{\xi_N}^P] \\ &= \sum_{\{\xi_j''\}} a(\{\xi_j''\}) \text{Asym}[\phi_{\xi_1}^P \otimes \cdots \otimes \phi_{\xi_N}^P] \end{aligned} \quad (5.1)$$

where $\xi_j' = (n_j, k_j + q \Delta k)$ and $\xi_j'' = (n_j, k_j - q \Delta k)$ with $\xi_j = (n_j, k_j)$ and $\Delta k = 2\pi/L_x$. Since $\Phi_{L_y}^{(N)}$ is an eigenvector of $T^{(N, y)}(q \Delta y)$ with the eigenvalue $\exp[i2\pi n/M]$ with an integer n , we have

$$a(\{\xi_j''\}) = a(\{\xi_j\}) \exp[i2\pi n/M] \quad (5.2)$$

Using the definition (4.4) of $U_{\pm, q}^{(\ell)}$, we have

$$\begin{aligned}
 & T^{(N, y)}(q \Delta y) \Psi_{\pm, L_y}^{(N)} \\
 &= \sum_{\{\xi_j\}} a(\{\xi_j\}) \exp \left[\pm i 2\pi \sum_{j=1}^N \tilde{m}(m_j)/\ell \right] \text{Asym}[\phi_{\xi'_1}^P \otimes \cdots \otimes \phi_{\xi'_N}^P] \\
 &= \sum_{\{\xi_j\}} a(\{\xi_j\}) \exp \left[\pm i 2\pi \sum_{j=1}^N \tilde{m}(m'_j)/\ell \right] \exp \left[\mp i 2\pi \sum_{s=1}^{q\ell} \hat{n}_s/\ell \right] \\
 &\quad \times \text{Asym}[\phi_{\xi'_1}^P \otimes \cdots \otimes \phi_{\xi'_N}^P] \\
 &= \sum_{\{\xi_j\}} a(\{\xi_j''\}) \exp \left[\pm i 2\pi \sum_{j=1}^N \tilde{m}(m_j)/\ell \right] \exp \left[\mp i 2\pi \sum_{s=-q+1}^{q(\ell-1)} \hat{n}_s/\ell \right] \\
 &\quad \times \text{Asym}[\phi_{\xi'_1}^P \otimes \cdots \otimes \phi_{\xi'_N}^P] \\
 &= e^{2\pi i n/M} \sum_{\{\xi_j\}} a(\{\xi_j\}) \exp \left[\pm i 2\pi \sum_{j=1}^N \frac{\tilde{m}(m_j)}{\ell} \right] \exp \left[\mp i 2\pi \sum_{s=-q+1}^{q(\ell-1)} \frac{\hat{n}_s}{\ell} \right] \\
 &\quad \times \text{Asym}[\phi_{\xi'_1}^P \otimes \cdots \otimes \phi_{\xi'_N}^P] \\
 &= e^{2\pi i n/M} \exp \left[\mp i \frac{2\pi}{\ell} \sum_{s=-q+1}^{q(\ell-1)} \hat{n}_s \right] \Psi_{\pm}^{(N)} \tag{5.3}
 \end{aligned}$$

where $k_j = 2\pi m_j/L_x$, $k'_j = 2\pi m'_j/L_x$, and we have used the relation (5.2).

Proof of Proposition 4.2. Following Tasaki,⁽³⁰⁾ we prove the statement. From (5.3), one has

$$\begin{aligned}
 & \langle \Phi_{L_y}^{(N)}, \Psi_{\pm, L_y}^{(N)} \rangle \\
 &= \langle T^{(N, y)}(q \Delta y) \Phi_{L_y}^{(N)}, T^{(N, y)}(q \Delta y) \Psi_{\pm, L_y}^{(N)} \rangle \\
 &= \left\langle \Phi_{L_y}^{(N)}, \exp \left[\mp i \frac{2\pi}{\ell} \sum_{s=-q+1}^{q(\ell-1)} \hat{n}_s \right] \Psi_{\pm, L_y}^{(N)} \right\rangle \\
 &= e^{\mp i 2\pi q v} \langle \Phi_{L_y}^{(N)}, \Psi_{\pm, L_y}^{(N)} \rangle \\
 &\quad + \left\langle \Phi_{L_y}^{(N)}, \left(\exp \left[\mp i \frac{2\pi}{\ell} \sum_{s=-q+1}^{q(\ell-1)} \hat{n}_s \right] - e^{\mp i 2\pi q v} \right) \Psi_{\pm, L_y}^{(N)} \right\rangle \tag{5.4}
 \end{aligned}$$

Using the Schwarz inequality, the second term in the last line is evaluated as

$$\begin{aligned}
& \left| \left\langle \Phi_{L_y}^{(N)}, \left(\exp \left[\mp i \frac{2\pi}{\ell} \sum_{s=-q+1}^{q(\ell-1)} \hat{n}_s \right] - e^{\mp i 2\pi q v} \right) \Psi_{\pm, L_y}^{(N)} \right\rangle \right|^2 \\
& \leq 4 \left\langle \Phi_{L_y}^{(N)}, \sin^2 \left[\pi \left(\frac{1}{\ell} \sum_{s=-q+1}^{q(\ell-1)} \hat{n}_s - qv \right) \right] \Phi_{L_y}^{(N)} \right\rangle \\
& \leq 4\pi^2 \left\langle \Phi_{L_y}^{(N)}, \left(\frac{1}{\ell} \sum_{s=-q+1}^{q(\ell-1)} \hat{n}_s - qv \right)^2 \Phi_{L_y}^{(N)} \right\rangle \\
& = 4\pi^2 \left[\frac{1}{\ell^2} \sum_{s,t} \langle \Phi_{L_y}^{(N)}, \hat{n}_s \hat{n}_t \Phi_{L_y}^{(N)} \rangle - (qv)^2 \right] \tag{5.5}
\end{aligned}$$

Here, for getting the last equality we have used the identity

$$\frac{1}{q} \sum_{s=j+1}^{j+q} \langle \Phi_{L_y}^{(N)}, \hat{n}_s \Phi_{L_y}^{(N)} \rangle = v \quad \text{for any lattice site } j \tag{5.6}$$

This is a consequence of the translational invariance of the state $\langle \Phi_{L_y}^{(N)}, (\dots) \Phi_{L_y}^{(N)} \rangle$ for the action $T^{(N,y)}(q \Delta y)$. From (5.4) and (5.5), one can show

$$\sin^2(\pi qv) |\omega(U_{\pm,1}^{(\ell)})|^2 \leq \pi^2 \left[\frac{1}{\ell^2} \sum_{s,t} \omega(\hat{n}_s \hat{n}_t) - (qv)^2 \right] \tag{5.7}$$

This right-hand side is nothing but the long range charge correlation which is vanishing in the limit $\ell \rightarrow \infty$ by the assumption (4.15). Therefore the statement of Proposition 4.2 has been proved. ■

6. ESTIMATE OF THE ENERGY GAP

In this section, we prove Lemma 4.1. For simplicity, we write $\Delta E^{(N)}$ by dropping the subscript L_y of $\Delta E_{L_y}^{(N)}$ of (4.7).

From the definition (4.7) of $\Delta E^{(N)}$, we have

$$\Delta E^{(N)} = \eta_{L_y}^{(N)}(H_{L_y}^{(N)}(n_{\max})) - \omega_{L_y}^{(N)}(H_{L_y}^{(N)}(n_{\max})) = \Delta E_W^{(N)} + \Delta E_U^{(N)} \tag{6.1}$$

with

$$\Delta E_W^{(N)} = \sum_{j=1}^N [\eta_{L_y}^{(N)}(W(x_j)) - \omega_{L_y}^{(N)}(W(x_j))] \tag{6.2}$$

and

$$\Delta E_U^{(N)} = \eta_{L_y}^{(N)}(U^{(N)}) - \omega_{L_y}^{(N)}(U^{(N)}) \quad (6.3)$$

In the following, we will estimate only $\Delta E_U^{(N)}$ because $\Delta E_W^{(N)}$ can be treated in a much easier way.

To begin with, we note that

$$\begin{aligned} \Delta E_U^{(N)} &= \sum_{\{\xi_j\}, \{\xi'_j\}} a^*(\{\xi_j\}) a(\{\xi'_j\}) \\ &\times \frac{1}{2} \left[\prod_{j=1}^N e^{i2\pi(\tilde{m}(m'_j) - \tilde{m}(m_j))/\ell} + \prod_{j=1}^N e^{-i2\pi(\tilde{m}(m'_j) - \tilde{m}(m_j))/\ell} - 2 \right] \\ &\times \langle \text{Asym}[\phi_{\xi_1}^{\text{P}} \otimes \cdots \otimes \phi_{\xi_N}^{\text{P}}], U^{(N)} \text{Asym}[\phi_{\xi'_1}^{\text{P}} \otimes \cdots \otimes \phi_{\xi'_N}^{\text{P}}] \rangle \end{aligned} \quad (6.4)$$

Here we notice that the contribution from $\{\xi_j\} = \{\xi'_j\}$ is vanishing, and that the matrix element for $U^{(N)}$ is vanishing if $\{\xi_j\}, \{\xi'_j\}$ differ by more than two pairs of single-body functions. Therefore $\Delta E_U^{(N)}$ can be written as

$$\Delta E_U^{(N)} = \Delta E_{\text{I}}^{(N)} + \Delta E_{\text{II}}^{(N)} \quad (6.5)$$

in terms of the two types of contributions, $\{\xi_j\}, \{\xi'_j\}$ differing by one pair of functions,

$$\begin{aligned} \Delta E_{\text{I}}^{(N)} &= \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{\xi'_s} a^*(\{\xi_j\}) a(\{\xi_1, \dots, \xi'_s, \dots, \xi_N\}) \\ &\times \left\{ \cos \left[\frac{2\pi}{\ell} (\tilde{m}(m'_s) - \tilde{m}(m_s)) \right] - 1 \right\} \\ &\times \langle \text{Asym}[\phi_{\xi_1}^{\text{P}} \otimes \cdots \otimes \phi_{\xi_N}^{\text{P}}], \\ &U^{(N)} \text{Asym}[\phi_{\xi_1}^{\text{P}} \otimes \cdots \otimes \phi_{\xi'_s}^{\text{P}} \otimes \cdots \otimes \phi_{\xi_N}^{\text{P}}] \rangle \end{aligned} \quad (6.6)$$

and $\{\xi_j\}, \{\xi'_j\}$ differing by two pairs of functions,

$$\begin{aligned} \Delta E_{\text{II}}^{(N)} &= \frac{1}{4} \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{t \neq s} \sum_{\xi'_s} \sum_{\xi'_t} a^*(\{\xi_j\}) a(\{\xi_1, \dots, \xi'_s, \dots, \xi'_t, \dots, \xi_N\}) \\ &\times \left\{ \cos \left[\frac{2\pi}{\ell} (\tilde{m}(m'_s) - \tilde{m}(m_s) + \tilde{m}(m'_t) - \tilde{m}(m_t)) \right] - 1 \right\} \\ &\times \langle \text{Asym}[\phi_{\xi_1}^{\text{P}} \otimes \cdots \otimes \phi_{\xi_N}^{\text{P}}], \\ &U^{(N)} \text{Asym}[\phi_{\xi_1}^{\text{P}} \otimes \cdots \otimes \phi_{\xi'_s}^{\text{P}} \otimes \cdots \otimes \phi_{\xi'_t}^{\text{P}} \otimes \cdots \otimes \phi_{\xi_N}^{\text{P}}] \rangle \end{aligned} \quad (6.7)$$

6.1. Estimate of $\Delta E_I^{(N)}$

We first treat $\Delta E_I^{(N)}$, and we will estimate $\Delta E_{II}^{(N)}$ in Section 6.2. To begin with, we decompose $\Delta E_I^{(N)}$ into the following two parts:

$$\begin{aligned} \Delta E_{I, <}^{(N)} &= \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{\xi'_s} a^*(\{\xi_j\}) a(\{\xi'_j\}) \chi(\text{dist}^{(m)}(m_s, m'_s) < \ell^\delta/2) \\ &\quad \times \left\{ \cos \left[\frac{2\pi}{\ell} (\tilde{m}(m'_s) - \tilde{m}(m_s)) \right] - 1 \right\} \\ &\quad \times \langle \text{Asym}[\phi_{\xi_1}^P \otimes \cdots \otimes \phi_{\xi_N}^P], \\ &\quad U^{(N)} \text{Asym}[\phi_{\xi_1}^P \otimes \cdots \otimes \phi_{\xi'_s}^P \otimes \cdots \otimes \phi_{\xi_N}^P] \rangle \end{aligned} \quad (6.8)$$

and

$$\begin{aligned} \Delta E_{I, \geq}^{(N)} &= \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{\xi'_s} a^*(\{\xi_j\}) a(\{\xi'_j\}) \chi(\text{dist}^{(m)}(m_s, m'_s) \geq \ell^\delta/2) \\ &\quad \times \left\{ \cos \left[\frac{2\pi}{\ell} (\tilde{m}(m'_s) - \tilde{m}(m_s)) \right] - 1 \right\} \\ &\quad \times \langle \text{Asym}[\phi_{\xi_1}^P \otimes \cdots \otimes \phi_{\xi_N}^P], \\ &\quad U^{(N)} \text{Asym}[\phi_{\xi_1}^P \otimes \cdots \otimes \phi_{\xi'_s}^P \otimes \cdots \otimes \phi_{\xi_N}^P] \rangle \end{aligned} \quad (6.9)$$

where $\delta \in (0, 1/4)$, and χ is the characteristic function given by

$$\chi(Q) = \begin{cases} 1 & \text{if } Q \text{ is true} \\ 0, & \text{otherwise} \end{cases} \quad (6.10)$$

and

$$\text{dist}^{(m)}(m_s, m'_s) := \min_{n \in \mathbf{Z}} \{|m_s - m'_s - nM|\} \quad (6.11)$$

Here we have written $\{\xi'_j\} = \{\xi_1, \xi_2, \dots, \xi_{s-1}, \xi'_s, \xi_{s+1}, \dots, \xi_N\}$. In the following, we fix δ to a number in the interval.

6.1.1. Estimate of $\Delta E_{I, <}^{(N)}$. As we will show in the following, we have a bound

$$|\Delta E_{I, <}^{(N)}| \leq 2\pi^2 q \ell^{(1)}(U^{(2)})(n_{\max} + 1)^3 \frac{\ell^{3\delta}}{\ell} \quad (6.12)$$

where $\mathcal{C}^{(1)}(U^{(2)})$ is a positive constant which depends only on¹⁹ the interaction $U^{(2)}$. Clearly $\Delta E_{1, <}^{(N)}$ is vanishing in the limit $\ell \rightarrow \infty$ because $\delta \in (0, 1/4)$.

In order to show the bound, we first evaluate the matrix element of $U^{(N)}$ in (6.8) as

$$\begin{aligned}
& |\langle \text{Asym}[\phi_{\xi_1}^{\mathbf{P}} \otimes \cdots \otimes \phi_{\xi_N}^{\mathbf{P}}], U^{(N)} \text{Asym}[\phi_{\xi_1}^{\mathbf{P}} \otimes \cdots \otimes \phi_{\xi'_s}^{\mathbf{P}} \otimes \cdots \otimes \phi_{\xi_N}^{\mathbf{P}}] \rangle | \\
& \leq \left| \sum_{\xi \in \{\xi_j\}} \int_S dx_1 dy_1 \int_S dx_2 dy_2 \phi_{\xi_s}^{\mathbf{P}^*}(\mathbf{r}_1) \phi_{\xi}^{\mathbf{P}^*}(\mathbf{r}_2) \right. \\
& \quad \times U^{(2)}(x_1 - x_2, y_1 - y_2) \phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_1) \phi_{\xi}^{\mathbf{P}}(\mathbf{r}_2) \left. \right| \\
& + \left| \sum_{\xi \in \{\xi_j\}} \int_S dx_1 dy_1 \int_S dx_2 dy_2 \phi_{\xi_s}^{\mathbf{P}^*}(\mathbf{r}_1) \phi_{\xi}^{\mathbf{P}^*}(\mathbf{r}_2) \right. \\
& \quad \times U^{(2)}(x_1 - x_2, y_1 - y_2) \phi_{\xi}^{\mathbf{P}}(\mathbf{r}_1) \phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_2) \left. \right| \tag{6.13}
\end{aligned}$$

Lemma 6.1. The following inequality is valid:

$$\begin{aligned}
& \sum_{\{\xi = (n, k) \mid n \leq n_{\max}\}} \int_S dx dy |\phi_{\xi}^{\mathbf{P}}(\mathbf{r})|^2 |U^{(2)}(x - x', y - y')| \\
& \leq \mathcal{C}^{(1)}(U^{(2)})(n_{\max} + 1) \quad \text{for any } x', y' \tag{6.14}
\end{aligned}$$

where $\mathcal{C}^{(1)}(U^{(2)})$ is the same constant as in (6.12).

The proof is given in Appendix B. Using Lemma 6.1, we have

$$\begin{aligned}
& \left| \sum_{\xi \in \{\xi_j\}} \int dx_1 dy_1 \int dx_2 dy_2 \phi_{\xi_s}^{\mathbf{P}^*}(\mathbf{r}_1) \phi_{\xi}^{\mathbf{P}^*}(\mathbf{r}_2) \right. \\
& \quad \times U^{(2)}(x_1 - x_2, y_1 - y_2) \phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_1) \phi_{\xi}^{\mathbf{P}}(\mathbf{r}_2) \left. \right| \\
& \leq \mathcal{C}^{(1)}(U^{(2)})(n_{\max} + 1) \int_S dx dy |\phi_{\xi}^{\mathbf{P}}(\mathbf{r})| |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r})| \tag{6.15}
\end{aligned}$$

for the first term in the right-hand side of (6.13). Here we have replaced the sum about $\{\xi_j\}$ with the sum about the whole ξ for getting the bound. From this bound, we obtain

¹⁹ For simplicity, we do not write a magnetic length ℓ_B dependence which is not of interest here.

$$\begin{aligned}
& |a^*(\{\xi_j\}) a(\{\xi'_j\})| \\
& \quad \times \left| \sum_{\xi \in \{\xi_j\}} \int dx_1 dy_1 \int dx_2 dy_2 \phi_{\xi_s}^{\mathbf{P}^*}(\mathbf{r}_1) \phi_{\xi}^{\mathbf{P}^*}(\mathbf{r}_2) \right. \\
& \quad \left. \times U^{(2)}(x_1 - x_2, y_1 - y_2) \phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_1) \phi_{\xi}^{\mathbf{P}}(\mathbf{r}_2) \right| \\
& \leq \frac{1}{2} \mathcal{C}^{(1)}(U^{(2)})(n_{\max} + 1) \left[|a(\{\xi_j\})|^2 \int_S dx dy |\phi_{\xi_s}^{\mathbf{P}}(\mathbf{r})|^2 \right. \\
& \quad \left. + |a(\{\xi'_j\})|^2 \int_S dx dy |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r})|^2 \right] \\
& = \frac{1}{2} \mathcal{C}^{(1)}(U^{(2)})(n_{\max} + 1) [|a(\{\xi_j\})|^2 + |a(\{\xi'_j\})|^2] \quad (6.16)
\end{aligned}$$

On the other hand, the second term in the right-hand side of (6.13) is evaluated as

$$\begin{aligned}
& \left| \int dx_1 dy_1 \int dx_2 dy_2 \phi_{\xi_s}^{\mathbf{P}^*}(\mathbf{r}_1) \phi_{\xi}^{\mathbf{P}^*}(\mathbf{r}_2) U^{(2)}(x_1 - x_2, y_1 - y_2) \phi_{\xi}^{\mathbf{P}}(\mathbf{r}_1) \phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_2) \right| \\
& \leq \sqrt{\int dx_1 dy_1 \int dx_2 dy_2 |\phi_{\xi_s}^{\mathbf{P}}(\mathbf{r}_1)|^2 |U^{(2)}(x_1 - x_2, y_1 - y_2)| |\phi_{\xi}^{\mathbf{P}}(\mathbf{r}_2)|^2} \\
& \quad \times \sqrt{\int dx_1 dy_1 \int dx_2 dy_2 |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_1)|^2 |U^{(2)}(x_1 - x_2, y_1 - y_2)| |\phi_{\xi}^{\mathbf{P}}(\mathbf{r}_2)|^2} \quad (6.17)
\end{aligned}$$

by using the Schwarz inequality. In the same way as in (6.16), we obtain

$$\begin{aligned}
& |a^*(\{\xi_j\}) a(\{\xi'_j\})| \left| \int dx_1 dy_1 \int dx_2 dy_2 \phi_{\xi_s}^{\mathbf{P}^*}(\mathbf{r}_1) \phi_{\xi}^{\mathbf{P}^*}(\mathbf{r}_2) \right. \\
& \quad \left. \times U^{(2)}(x_1 - x_2, y_1 - y_2) \phi_{\xi}^{\mathbf{P}}(\mathbf{r}_1) \phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_2) \right| \\
& \leq \frac{1}{2} |a^*(\{\xi_j\})|^2 \int dx_1 dy_1 \int dx_2 dy_2 |\phi_{\xi_s}^{\mathbf{P}}(\mathbf{r}_1)|^2 \\
& \quad \times |U^{(2)}(x_1 - x_2, y_1 - y_2)| |\phi_{\xi}^{\mathbf{P}}(\mathbf{r}_2)|^2 \\
& \quad + \frac{1}{2} |a^*(\{\xi'_j\})|^2 \int dx_1 dy_1 \int dx_2 dy_2 |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_1)|^2 \\
& \quad \times |U^{(2)}(x_1 - x_2, y_1 - y_2)| |\phi_{\xi}^{\mathbf{P}}(\mathbf{r}_2)|^2 \quad (6.18)
\end{aligned}$$

Taking the sum over $\zeta \in \{\zeta_j\}$ and using Lemma 6.1 in the same way, we have

$$\begin{aligned}
 & |a^*(\{\zeta_j\}) a(\{\zeta'_j\})| \\
 & \quad \times \left| \sum_{\xi} \int dx_1 dy_1 \int dx_2 dy_2 \phi_{\xi_s}^{\mathbf{P}^*}(\mathbf{r}_1) \phi_{\xi}^{\mathbf{P}^*}(\mathbf{r}_2) \right. \\
 & \quad \left. \times U^{(2)}(x_1 - x_2, y_1 - y_2) \phi_{\xi}^{\mathbf{P}}(\mathbf{r}_1) \phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_2) \right| \\
 & \leq \frac{1}{2} \mathcal{C}^{(1)}(U^{(2)})(n_{\max} + 1) [|a^*(\{\zeta_j\})|^2 + |a(\{\zeta'_j\})|^2] \quad (6.19)
 \end{aligned}$$

Combining (6.13), (6.16) and (6.19), we obtain

$$\begin{aligned}
 & |a^*(\{\zeta_j\}) a(\{\zeta'_j\}) \langle \text{Asym}[\phi_{\xi_1}^{\mathbf{P}} \otimes \cdots \otimes \phi_{\xi_N}^{\mathbf{P}}], U^{(N)} \\
 & \quad \times \text{Asym}[\phi_{\xi'_1}^{\mathbf{P}} \otimes \cdots \otimes \phi_{\xi'_s}^{\mathbf{P}} \otimes \cdots \otimes \phi_{\xi'_N}^{\mathbf{P}}] \rangle| \\
 & \leq \mathcal{C}^{(1)}(U^{(2)})(n_{\max} + 1) [|a(\{\zeta_j\})|^2 + |a(\{\zeta'_j\})|^2] \quad (6.20)
 \end{aligned}$$

By using this inequality, $\Delta E_{\mathbf{I}_<}^{(N)}$ of (6.8) is evaluated as

$$\begin{aligned}
 |\Delta E_{\mathbf{I}_<}^{(N)}| & \leq 2 \mathcal{C}^{(1)}(U^{(2)})(n_{\max} + 1) \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{\xi'_s} \chi(\text{dist}^{(m)}(m_s, m'_s) < \ell^\delta/2) \\
 & \quad \times \left\{ 1 - \cos \left[\frac{2\pi}{\ell} (\tilde{m}(m'_s) - \tilde{m}(m_s)) \right] \right\} |a(\{\xi_j\})|^2 \quad (6.21)
 \end{aligned}$$

Note that

$$\begin{aligned}
 & 1 - \cos \left[\frac{2\pi}{\ell} (\tilde{m}(m'_s) - \tilde{m}(m_s)) \right] \\
 & = \left\{ 1 - \cos \left[\frac{2\pi}{\ell} (\tilde{m}(m'_s) - \tilde{m}(m_s)) \right] \right\} \\
 & \quad \times \{ \chi(1 \leq m_s \leq q\ell) + [1 - \chi(1 \leq m_s \leq q\ell)] \chi(1 \leq m'_s \leq q\ell) \} \quad (6.22)
 \end{aligned}$$

from the definition (4.5) of $\tilde{m}(\dots)$. Using this identity, we have

$$\begin{aligned}
 |\Delta E_{\mathbf{I}_<}^{(N)}| & \leq 2 \mathcal{C}^{(1)}(U^{(2)})(n_{\max} + 1) \\
 & \quad \times \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{\xi'_s} \{ \chi(1 \leq m_s \leq q\ell) \\
 & \quad + [1 - \chi(1 \leq m_s \leq q\ell)] \chi(1 \leq m'_s \leq q\ell) \}
 \end{aligned}$$

$$\begin{aligned}
& \times \chi(\text{dist}^{(m)}(m_s, m'_s) < \ell^\delta/2) \left\{ 1 - \cos \left[\frac{2\pi}{\ell} (\tilde{m}(m'_s) - \tilde{m}(m_s)) \right] \right\} \\
& \times |a(\{\xi_j\})|^2 \\
& \leq \mathcal{C}^{(1)}(U^{(2)})(n_{\max} + 1) \\
& \times \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{\xi'_s} \{ \chi(1 \leq m_s \leq q\ell) \\
& + [1 - \chi(1 \leq m_s \leq q\ell)] \chi(1 \leq m'_s \leq q\ell) \} \\
& \times \chi(\text{dist}^{(m)}(m_s, m'_s) < \ell^\delta/2) |a(\{\xi_j\})|^2 \frac{\pi^2 \ell^{2\delta}}{\ell^2} \tag{6.23}
\end{aligned}$$

Note that

$$\begin{aligned}
& \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{\xi'_s} \chi(1 \leq m_s \leq q\ell) \chi(\text{dist}^{(m)}(m_s, m'_s) < \ell^\delta/2) |a(\{\xi_j\})|^2 \\
& \leq \sum_{\{\xi_j\}} \sum_{s=1}^N \chi(1 \leq m_s \leq q\ell) (n_{\max} + 1) \ell^\delta |a(\{\xi_j\})|^2 \\
& \leq \sum_{\{\xi_j\}} (n_{\max} + 1)^2 q\ell \ell^\delta |a(\{\xi_j\})|^2 = (n_{\max} + 1)^2 q\ell \ell^\delta \tag{6.24}
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{\xi'_s} [1 - \chi(1 \leq m_s \leq q\ell)] \chi(1 \leq m'_s \leq q\ell) \\
& \quad \times \chi(\text{dist}^{(m)}(m_s, m'_s) < \ell^\delta/2) |a(\{\xi_j\})|^2 \\
& \leq \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{\xi'_s} \chi(-\ell^\delta/2 + 1 \leq m_s \leq 0 \text{ or } q\ell + 1 \leq m_s \leq q\ell + \ell^\delta/2 - 1) \\
& \quad \times \chi(\text{dist}^{(m)}(m_s, m'_s) < \ell^\delta/2) |a(\{\xi_j\})|^2 \\
& \leq \sum_{\{\xi_j\}} \sum_{s=1}^N \chi(-\ell^\delta/2 + 1 \leq m_s \leq 0 \text{ or } q\ell + 1 \leq m_s \leq q\ell + \ell^\delta/2 - 1) \\
& \quad \times (n_{\max} + 1) \ell^\delta |a(\{\xi_j\})|^2 \\
& \leq \sum_{\{\xi_j\}} (n_{\max} + 1)^2 \ell^{2\delta} |a(\{\xi_j\})|^2 = (n_{\max} + 1)^2 \ell^{2\delta} \tag{6.25}
\end{aligned}$$

Substituting these into (6.23), we obtain

$$\begin{aligned} |\Delta E_{I, <}^{(N)}| &\leq \pi^2 \mathcal{C}^{(1)}(U^{(2)})(n_{\max} + 1)^3 \left(\frac{q\ell^{3\delta}}{\ell} + \frac{\ell^{4\delta}}{\ell^2} \right) \\ &\leq 2\pi^2 q \mathcal{C}^{(1)}(U^{(2)})(n_{\max} + 1)^3 \frac{\ell^{3\delta}}{\ell} \end{aligned} \quad (6.26)$$

6.1.2. Estimate of $\Delta E_{I, \geq}^{(N)}$. The main results in this subsection are summarized in Lemma 6.2 below. The results include important information about the cutoff dependence of the size δy of the local perturbation in the y direction.

For proceeding to this lemma, we make preparations. Using (6.13), Lemma 6.1 and (6.22) in the same way as in the preceding Section 6.1.1, we can evaluate $\Delta E_{I, \geq}^{(N)}$ of (6.9) as

$$|\Delta E_{I, \geq}^{(N)}| \leq 2\mathcal{C}^{(1)}(U^{(2)})(n_{\max} + 1) \Delta E_{I, \geq, 1}^{(N)} + 2\Delta E_{I, \geq, 2}^{(N)} \quad (6.27)$$

where

$$\begin{aligned} \Delta E_{I, \geq, 1}^{(N)} &= \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{\xi'_s} [|a(\{\xi_j\})|^2 + |a(\{\xi'_j\})|^2] \chi(1 \leq m_s \leq q\ell) \\ &\quad \times \chi(\text{dist}^{(m)}(m_s, m'_s) \geq \ell^\delta/2) \int_S dx dy |\phi_{\xi'_s}^P(\mathbf{r})| |\phi_{\xi_s}^P(\mathbf{r})| \end{aligned} \quad (6.28)$$

and

$$\begin{aligned} \Delta E_{I, \geq, 2}^{(N)} &= \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{\xi'_s} [|a(\{\xi_j\})|^2 + |a(\{\xi'_j\})|^2] \\ &\quad \times \chi(1 \leq m_s \leq q\ell) \chi(\text{dist}^{(m)}(m_s, m'_s) \geq \ell^\delta/2) \\ &\quad \times \sum_{\xi \in \{\xi_j\}} \int dx_1 dy_1 \int dx_2 dy_2 |\phi_{\xi}^P(\mathbf{r}_1)| |\phi_{\xi}^P(\mathbf{r}_1)| \\ &\quad \times |U^{(2)}(x_1 - x_2, y_1 - y_2)| |\phi_{\xi'_s}^P(\mathbf{r}_2)| |\phi_{\xi}^P(\mathbf{r}_2)| \end{aligned} \quad (6.29)$$

Lemma 6.2. Suppose that $\pi(\ell^\delta/4 - 1)\ell_B/L_x > n_{\max}$ and $L_y > 32n_{\max}\ell_B$. Then the following two bounds are valid:

$$\Delta E_{I, \geq, 1}^{(N)} \leq 2(n_{\max} + 1) q\ell \varepsilon^{(1)}(\ell^\delta/2 - 1, n_{\max}, L_x, L_y) \quad (6.30)$$

and

$$\begin{aligned} \Delta E_{\mathbf{I}, \geq 2}^{(N)} &\leq 4 \|U^{(2)}\|_{\infty} (n_{\max} + 1) q \ell \varepsilon^{(1)}(\ell^{\delta}/4 - 1, n_{\max}, L_x, L_y) \\ &\quad \times [2(n_{\max} + 1) \ell^{\delta} + \varepsilon^{(1)}(\ell^{\delta} - 1, n_{\max}, L_x, L_y)] \end{aligned} \quad (6.31)$$

where

$$\begin{aligned} &\varepsilon^{(1)}(\Delta \ell, n_{\max}, L_x, L_y) \\ &:= \mathcal{C}^{(2)}(n_{\max}) \frac{L_x}{\ell_B} \exp \left[- \left(\frac{\pi \ell_B}{L_x} \Delta \ell - n_{\max} \right)^2 \right] \\ &\quad + \mathcal{C}^{(3)}(n_{\max}) \frac{L_x L_y}{\ell_B^2} \exp \left[- \frac{L_y^2}{32 \ell_B^2} \left(1 - \frac{32 n_{\max} \ell_B}{L_y} \right)^2 \right] \end{aligned} \quad (6.32)$$

Here the constants $\mathcal{C}^{(2)}(n_{\max})$ and $\mathcal{C}^{(3)}(n_{\max})$ depend on the energy cutoff n_{\max} only.

Immediately, we have

$$\lim_{\ell \rightarrow \infty} \lim_{L_y \rightarrow \infty} \Delta E_{\mathbf{I}, \geq 1}^{(N)} = 0, \quad \lim_{\ell \rightarrow \infty} \lim_{L_y \rightarrow \infty} \Delta E_{\mathbf{I}, \geq 2}^{(N)} = 0 \quad (6.33)$$

for a fixed L_x . Clearly these with (6.27) yield

$$\lim_{\ell \rightarrow \infty} \lim_{L_y \rightarrow \infty} \Delta E_{\mathbf{I}, \geq}^{(N)} = 0 \quad (6.34)$$

for a fixed L_x . We remark that the size δy of the local perturbation in the y direction strongly depends on the width L_x of the strip and the energy cutoff n_{\max} . To see this, we note that the size δy is given by $\delta y = \ell \Delta y$, where Δy is the lattice constant given by (3.13). From the lemma, the number ℓ must at least satisfy $\ell_B \ell^{\delta} \sim L_x n_{\max}$. Then we have

$$\delta y \propto (L_x)^{1/\delta - 1} (n_{\max})^{1/\delta} = (L_x)^{3+\varepsilon} (n_{\max})^{4+\varepsilon} \quad (6.35)$$

where we have taken ε to be a small positive number by using $\delta \in (0, 1/4)$. This cutoff dependence is not a desired one. We believe that the size δy does not depend on L_x, n_{\max} in much better evaluation for the low energy excitations.

First let us show the bound (6.30). We can rewrite $\Delta E_{\mathbf{I}, \geq 1}^{(N)}$ as

$$\begin{aligned} \Delta E_{\mathbf{I}, \geq 1}^{(N)} &= \sum_{\{\xi_j\}} \sum_{s=1}^N \chi(1 \leq m_s \leq q\ell) |a(\{\xi_j\})|^2 \\ &\quad \times \sum_{\xi'_s} \chi(\text{dist}^{(m)}(m_s, m'_s) \geq \ell^\delta/2) \int_S dx dy |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r})| |\phi_{\xi_s}^{\mathbf{P}}(\mathbf{r})| \\ &\quad + \sum_{\{\xi'_j\}} |a(\{\xi'_j\})|^2 \sum_{s=1}^N \sum_{\xi_s} \chi(1 \leq m_s \leq q\ell) \\ &\quad \times \chi(\text{dist}^{(m)}(m_s, m'_s) \geq \ell^\delta/2) \int_S dx dy |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r})| |\phi_{\xi_s}^{\mathbf{P}}(\mathbf{r})| \quad (6.36) \end{aligned}$$

Note that

$$\begin{aligned} &\sum_{s=1}^N \sum_{\xi_s} \chi(1 \leq m_s \leq q\ell) \chi(\text{dist}^{(m)}(m_s, m'_s) \geq \ell^\delta/2) \int_S dx dy |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r})| |\phi_{\xi_s}^{\mathbf{P}}(\mathbf{r})| \\ &\leq \sum_{\xi} \chi(1 \leq m \leq q\ell) \sum_{s=1}^N \chi(\text{dist}^{(m)}(m, m'_s) \geq \ell^\delta/2) \int_S dx dy |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r})| |\phi_{\xi}^{\mathbf{P}}(\mathbf{r})| \quad (6.37) \end{aligned}$$

where the first sum in the right-hand side is taken over all the states $\xi = (n, k)$ with $k = 2\pi m/L_x$.

We note that the Hermite polynomial H_n in the functions $\phi_{n,k}^{\mathbf{P}}$ satisfies

$$|H_n(\zeta)| \leq c_n e^{\beta_n |\zeta|} \quad \text{for } \zeta \in \mathbf{R} \quad (6.38)$$

where the positive constants c_n and β_n depend only on the number n . The well-known values are given by

$$c_n = \begin{cases} (n-1)!! & \text{for } n = \text{even} \\ n!! & \text{for } n = \text{odd} \end{cases} \quad (6.39)$$

and

$$\beta_n = \begin{cases} \sqrt{2n} & \text{for } n = \text{even} \\ \sqrt{2(n-1)} & \text{for } n = \text{odd} \end{cases} \quad (6.40)$$

Here $(2n-1)!! = (2n-1)(2n-3)\dots 3 \cdot 1$ with $(-1)!! = 1$. Using the bound (6.38), we can get the following lemma:

Lemma 6.3. Let ϕ_{ξ}^{P} be an eigenvector (3.17) of the Hamiltonian (3.1) with the periodic boundary conditions (3.9) and with quantum numbers $\xi = (n, k) = (n, 2\pi m/L_x)$, and let $\pi(\Delta\ell - 1)\ell_B/L_x > n_{\max}$ and $L_y > 32n_{\max}\ell_B$. Then the following bound is valid:

$$\sum_{\xi'} \chi(\text{dist}^{(m)}(m, m') \geq \Delta\ell) \int_{\mathcal{S}} dx dy |\phi_{\xi}^{\text{P}}(\mathbf{r})| |\phi_{\xi'}^{\text{P}}(\mathbf{r})| \leq \varepsilon^{(1)}(\Delta\ell - 1, n_{\max}, L_x, L_y) \quad (6.41)$$

where the sum is over all the states $\xi' = (n', k')$ with $k' = 2\pi m'/L_x$, and $\varepsilon^{(1)}$ is given by (6.32).

The proof is given in Appendix C. Combining (6.36), (6.37) and Lemma 6.3, we obtain the desired bound (6.30).

Next consider $\Delta E_{\mathbf{I}, \geq 2}^{(N)}$ of (6.29), and we shall show the bound (6.31). Using the inequality

$$\begin{aligned} & \chi\left(\text{dist}^{(m)}(m_s, m'_s) \geq \frac{\ell^\delta}{2}\right) \\ & \leq \chi\left(\text{dist}^{(m)}(m_s, m) \geq \frac{\ell^\delta}{4}\right) + \chi\left(\text{dist}^{(m)}(m'_s, m) \geq \frac{\ell^\delta}{4}\right) \end{aligned} \quad (6.42)$$

we have

$$\Delta E_{\mathbf{I}, \geq 2}^{(N)} \leq \|U^{(2)}\|_{\infty} \sum_{i=1}^4 \Delta E_{\mathbf{I}, \geq 2, i}^{(N)} \quad (6.43)$$

where

$$\begin{aligned} \Delta E_{\mathbf{I}, \geq 2, 1}^{(N)} &= \sum_{\{\xi_j\}} \sum_{s=1}^N |a(\{\xi_j\})|^2 \chi(1 \leq m_s \leq q\ell) \\ & \times \sum_{\xi} \chi(\text{dist}^{(m)}(m_s, m) \geq \ell^\delta/4) \int dx_1 dy_1 |\phi_{\xi}^{\text{P}}(\mathbf{r}_1)| |\phi_{\xi}^{\text{P}}(\mathbf{r}_1)| \\ & \times \sum_{\xi'} \int dx_2 dy_2 |\phi_{\xi'}^{\text{P}}(\mathbf{r}_2)| |\phi_{\xi}^{\text{P}}(\mathbf{r}_2)| \end{aligned} \quad (6.44)$$

$$\begin{aligned}
\Delta E_{1, \geq 2, 2}^{(N)} &= \sum_{\{\xi_j\}} \sum_{s=1}^N |a(\{\xi_j\})|^2 \chi(1 \leq m_s \leq q\ell) \\
&\quad \times \sum_{\xi} \int dx_1 dy_1 |\phi_{\xi}^{\mathbf{P}}(\mathbf{r}_1)| |\phi_{\xi}^{\mathbf{P}}(\mathbf{r}_1)| \\
&\quad \times \sum_{\xi'_s} \chi(\text{dist}^{(m)}(m'_s, m) \geq \ell^\delta/4) \int dx_2 dy_2 |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_2)| |\phi_{\xi}^{\mathbf{P}}(\mathbf{r}_2)|
\end{aligned} \tag{6.45}$$

$$\begin{aligned}
\Delta E_{1, \geq 2, 3}^{(N)} &= \sum_{\{\xi'_j\}} |a(\{\xi'_j\})|^2 \sum_{\xi''} \chi(1 \leq m'' \leq q\ell) \\
&\quad \times \sum_{\xi} \chi(\text{dist}^{(m)}(m'', m) \geq \ell^\delta/4) \int dx_1 dy_1 |\phi_{\xi''}^{\mathbf{P}}(\mathbf{r}_1)| |\phi_{\xi}^{\mathbf{P}}(\mathbf{r}_1)| \\
&\quad \times \sum_{s=1}^N \int dx_2 dy_2 |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_2)| |\phi_{\xi}^{\mathbf{P}}(\mathbf{r}_2)|
\end{aligned} \tag{6.46}$$

and

$$\begin{aligned}
\Delta E_{1, \geq 2, 4}^{(N)} &= \sum_{\{\xi'_j\}} |a(\{\xi'_j\})|^2 \sum_{\xi''} \chi(1 \leq m'' \leq q\ell) \\
&\quad \times \sum_{\xi} \int dx_1 dy_1 |\phi_{\xi''}^{\mathbf{P}}(\mathbf{r}_1)| |\phi_{\xi}^{\mathbf{P}}(\mathbf{r}_1)| \\
&\quad \times \sum_{s=1}^N \chi(\text{dist}^{(m)}(m'_s, m) \geq \ell^\delta/4) \int dx_2 dy_2 |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_2)| |\phi_{\xi}^{\mathbf{P}}(\mathbf{r}_2)|
\end{aligned} \tag{6.47}$$

We note that, from Lemma 6.3,

$$\begin{aligned}
&\sum_{\xi'} \int_S dx dy |\phi_{\xi'}^{\mathbf{P}}(\mathbf{r})| |\phi_{\xi}^{\mathbf{P}}(\mathbf{r})| \\
&= \sum_{\xi'} [\chi(\text{dist}^{(m)}(m, m') < \ell^\delta) + \chi(\text{dist}^{(m)}(m, m') \geq \ell^\delta)] \\
&\quad \times \int_S dx dy |\phi_{\xi'}^{\mathbf{P}}(\mathbf{r})| |\phi_{\xi}^{\mathbf{P}}(\mathbf{r})| \\
&\leq 2(n_{\max} + 1) \ell^\delta + \varepsilon^{(1)}(\ell^\delta - 1, n_{\max}, L_x, L_y)
\end{aligned} \tag{6.48}$$

Using this inequality and Lemma 6.3 again, we obtain

$$\begin{aligned} \Delta E_{1, \geq 2, i}^{(N)} &\leq (n_{\max} + 1) q \ell \varepsilon^{(1)} (\ell^\delta / 4 - 1, n_{\max}, L_x, L_y) \\ &\quad \times [2(n_{\max} + 1) \ell^\delta + \varepsilon^{(1)} (\ell^\delta - 1, n_{\max}, L_x, L_y)] \end{aligned} \quad (6.49)$$

for $i = 1, 2, 3, 4$. Substituting these into (6.43), we get the desired bound (6.31).

6.2. Estimate of $\Delta E_{\text{II}}^{(N)}$

For $\Delta E_{\text{II}}^{(N)}$ of (6.7), one can easily obtain

$$\begin{aligned} |\Delta E_{\text{II}}^{(N)}| &\leq \frac{1}{2} \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{t \neq s} \sum_{\xi'_s} \sum_{\xi'_t} |a^*(\{\xi_j\}) a(\{\xi''_j\})| \\ &\quad \times \left\{ 1 - \cos \left[\frac{2\pi}{\ell} (\tilde{m}(m'_s) - \tilde{m}(m_s) + \tilde{m}(m'_t) - \tilde{m}(m_t)) \right] \right\} \\ &\quad \times \left| \int dx_1 dy_1 \int dx_2 dy_2 \phi_{\xi'_s}^{\text{P}*}(\mathbf{r}_1) \phi_{\xi'_t}^{\text{P}*}(\mathbf{r}_2) \right. \\ &\quad \left. \times U^{(2)}(x_1 - x_2, y_1 - y_2) \phi_{\xi'_s}^{\text{P}}(\mathbf{r}_1) \phi_{\xi'_t}^{\text{P}}(\mathbf{r}_2) \right| \end{aligned} \quad (6.50)$$

where we have written

$$\{\xi''_j\} = \{\xi_1, \dots, \xi_{s-1}, \xi'_s, \xi_{s+1}, \dots, \xi_{t-1}, \xi'_t, \xi_{t+1}, \dots, \xi_N\} \quad (6.51)$$

The right-hand side of (6.50) can be decomposed into the following two parts:

$$\begin{aligned} \Delta \tilde{E}_{\text{II}, <}^{(N)} &= \frac{1}{2} \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{t \neq s} \sum_{\xi'_s} \sum_{\xi'_t} |a^*(\{\xi_j\}) a(\{\xi''_j\})| \\ &\quad \times \left\{ 1 - \cos \left[\frac{2\pi}{\ell} (\tilde{m}(m'_s) - \tilde{m}(m_s) + \tilde{m}(m'_t) - \tilde{m}(m_t)) \right] \right\} \\ &\quad \times \chi(\text{dist}^{(m)}(m_s, m'_s) < \ell^\delta / 2) \chi(\text{dist}^{(m)}(m_t, m'_t) < \ell^\delta / 2) \\ &\quad \times \left| \int dx_1 dy_1 \int dx_2 dy_2 \phi_{\xi'_s}^{\text{P}*}(\mathbf{r}_1) \phi_{\xi'_t}^{\text{P}*}(\mathbf{r}_2) \right. \\ &\quad \left. \times U^{(2)}(x_1 - x_2, y_1 - y_2) \phi_{\xi'_s}^{\text{P}}(\mathbf{r}_1) \phi_{\xi'_t}^{\text{P}}(\mathbf{r}_2) \right| \end{aligned} \quad (6.52)$$

and

$$\begin{aligned}
\Delta \tilde{E}_{\Pi, >}^{(N)} &= \frac{1}{2} \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{t \neq s} \sum_{\xi'_s} \sum_{\xi'_t} |a^*(\{\xi_j\}) a(\{\xi'_j\})| \\
&\times \left\{ 1 - \cos \left[\frac{2\pi}{\ell} (\tilde{m}(m'_s) - \tilde{m}(m_s) + \tilde{m}(m'_t) - \tilde{m}(m_t)) \right] \right\} \\
&\times [1 - \chi(\text{dist}^{(m)}(m_s, m'_s) < \ell^\delta/2) \chi(\text{dist}^{(m)}(m_t, m'_t) < \ell^\delta/2)] \\
&\times \left| \int dx_1 dy_1 \int dx_2 dy_2 \phi_{\xi'_s}^{\text{P}*}(\mathbf{r}_1) \phi_{\xi'_t}^{\text{P}*}(\mathbf{r}_2) \right. \\
&\times U^{(2)}(x_1 - x_2, y_1 - y_2) \phi_{\xi'_s}^{\text{P}}(\mathbf{r}_1) \phi_{\xi'_t}^{\text{P}}(\mathbf{r}_2) \left. \right| \quad (6.53)
\end{aligned}$$

6.2.1. Estimate of $\Delta \tilde{E}_{\Pi, <}^{(N)}$. As we will show in the following, we obtain

$$\Delta \tilde{E}_{\Pi, <}^{(N)} \leq 4\pi^2 q(n_{\max} + 1)^4 \mathcal{C}^{(1)}(U^{(2)}) \frac{\ell^{4\delta}}{\ell} \quad (6.54)$$

for $\Delta \tilde{E}_{\Pi, <}^{(N)}$ of (6.52). Since $\delta \in (0, 1/4)$, $\Delta \tilde{E}_{\Pi, <}^{(N)}$ is vanishing in the limit $\ell \rightarrow \infty$.

In order to show the bound (6.54), consider first the integral in the right-hand side of (6.52). Using the Schwarz inequality, we have

$$\begin{aligned}
&\left| \int dx_1 dy_1 \int dx_2 dy_2 \phi_{\xi'_s}^{\text{P}*}(\mathbf{r}_1) \phi_{\xi'_t}^{\text{P}*}(\mathbf{r}_2) U^{(2)}(x_1 - x_2, y_1 - y_2) \phi_{\xi'_s}^{\text{P}}(\mathbf{r}_1) \phi_{\xi'_t}^{\text{P}}(\mathbf{r}_2) \right| \\
&\leq \sqrt{\int dx_1 dy_1 \int dx_2 dy_2 |\phi_{\xi'_s}^{\text{P}}(\mathbf{r}_1)|^2 |U^{(2)}(x_1 - x_2, y_1 - y_2)| |\phi_{\xi'_t}^{\text{P}}(\mathbf{r}_2)|^2} \\
&\times \sqrt{\int dx_1 dy_1 \int dx_2 dy_2 |\phi_{\xi'_s}^{\text{P}*}(\mathbf{r}_1)|^2 |U^{(2)}(x_1 - x_2, y_1 - y_2)| |\phi_{\xi'_t}^{\text{P}*}(\mathbf{r}_2)|^2} \quad (6.55)
\end{aligned}$$

Thereby

$$\begin{aligned}
&\left| a^*(\{\xi_j\}) a(\{\xi'_j\}) \int dx_1 dy_1 \int dx_2 dy_2 \phi_{\xi'_s}^{\text{P}*}(\mathbf{r}_1) \right. \\
&\times \phi_{\xi'_t}^{\text{P}*}(\mathbf{r}_2) U^{(2)}(x_1 - x_2, y_1 - y_2) \phi_{\xi'_s}^{\text{P}}(\mathbf{r}_1) \phi_{\xi'_t}^{\text{P}}(\mathbf{r}_2) \left. \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2} \left[|a(\{\xi_j\})|^2 \int dx_1 dy_1 \int dx_2 d_2 |\phi_{\xi_s}^{\mathbf{P}}(\mathbf{r}_1)|^2 \right. \\
&\quad \times |U^{(2)}(x_1 - x_2, y_1 - y_2)| |\phi_{\xi_t}^{\mathbf{P}}(\mathbf{r}_2)|^2 \\
&\quad + |a(\{\xi_j''\})|^2 \int dx_1 dy_1 \int dx_2 d_2 |\phi_{\xi_s}^{\mathbf{P}}(\mathbf{r}_1)|^2 \\
&\quad \left. \times |U^{(2)}(x_1 - x_2, y_1 - y_2)| |\phi_{\xi_t}^{\mathbf{P}}(\mathbf{r}_2)|^2 \right] \quad (6.56)
\end{aligned}$$

Substituting this into (6.52), we get

$$\begin{aligned}
\Delta \tilde{E}_{\text{II}, <}^{(N)} &\leq \frac{1}{2} \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{t \neq s} \sum_{\xi_s'} \sum_{\xi_t'} |a(\{\xi_j\})|^2 \\
&\quad \times \left\{ 1 - \cos \left[\frac{2\pi}{\ell} (\tilde{m}(m'_s) - \tilde{m}(m_s) + \tilde{m}(m'_t) - \tilde{m}(m_t)) \right] \right\} \\
&\quad \times \chi(\text{dist}^{(m)}(m_s, m'_s) < \ell^\delta/2) \chi(\text{dist}^{(m)}(m_t, m'_t) < \ell^\delta/2) \\
&\quad \times \int dx_1 dy_1 \int dx_2 dy_2 |\phi_{\xi_s}^{\mathbf{P}}(\mathbf{r}_1)|^2 |U^{(2)}(x_1 - x_2, y_1 - y_2)| |\phi_{\xi_t}^{\mathbf{P}}(\mathbf{r}_2)|^2 \quad (6.57)
\end{aligned}$$

Here, if all of m_s, m'_s, m_t, m'_t are not in the interval $[1, q\ell]$, then the corresponding contributions are vanishing from the definitions (4.5) of $\tilde{m}(\dots)$ and the cosine function. From this observation, we have

$$\begin{aligned}
\Delta \tilde{E}_{\text{II}, <}^{(N)} &\leq \frac{1}{2} \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{t \neq s} \sum_{\xi_s'} \sum_{\xi_t'} |a(\{\xi_j\})|^2 \\
&\quad \times \left\{ 1 - \cos \left[\frac{2\pi}{\ell} (\tilde{m}(m'_s) - \tilde{m}(m_s) + \tilde{m}(m'_t) - \tilde{m}(m_t)) \right] \right\} \\
&\quad \times \chi(\text{dist}^{(m)}(m_s, m'_s) < \ell^\delta/2) \chi(\text{dist}^{(m)}(m_t, m'_t) < \ell^\delta/2) \\
&\quad \times \{ \chi(1 \leq m_s \leq q\ell) + [1 - \chi(1 \leq m_s \leq q\ell)] \chi(1 \leq m_t \leq q\ell) \\
&\quad + [1 - \chi(1 \leq m_s \leq q\ell)] [1 - \chi(1 \leq m_t \leq q\ell)] \\
&\quad \times \chi(1 \leq m'_s \leq q\ell) \\
&\quad + [1 - \chi(1 \leq m_s \leq q\ell)] [1 - \chi(1 \leq m_t \leq q\ell)] \\
&\quad \times [1 - \chi(1 \leq m'_s \leq q\ell)] \chi(1 \leq m'_t \leq q\ell) \} \int dx_1 dy_1 \int dx_2 \\
&\quad \times dy_2 |\phi_{\xi_s}^{\mathbf{P}}(\mathbf{r}_1)|^2 |U^{(2)}(x_1 - x_2, y_1 - y_2)| |\phi_{\xi_t}^{\mathbf{P}}(\mathbf{r}_2)|^2 \quad (6.58)
\end{aligned}$$

Note that

$$\begin{aligned}
& \frac{1}{2} \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{t \neq s} \sum_{\xi'_s} \sum_{\xi'_t} |a(\{\xi_j\})|^2 \\
& \quad \times \left\{ 1 - \cos \left[\frac{2\pi}{\ell} (\tilde{m}(m'_s) - \tilde{m}(m_s) + \tilde{m}(m'_t) - \tilde{m}(m_t)) \right] \right\} \\
& \quad \times \chi(\text{dist}^{(m)}(m_s, m'_s) < \ell^\delta/2) \chi(\text{dist}^{(m)}(m_t, m'_t) < \ell^\delta/2) \\
& \quad \times \chi(1 \leq m_s \leq q\ell) \int dx_1 dy_1 \int dx_2 dy_2 |\phi_{\xi_s}^{\mathbf{P}}(\mathbf{r}_1)|^2 \\
& \quad \times |U^{(2)}(x_1 - x_2, y_1 - y_2)| |\phi_{\xi'_t}^{\mathbf{P}}(\mathbf{r}_2)|^2 \\
& \leq \pi^2 \frac{\ell^{2\delta}}{\ell^2} \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{t \neq s} \sum_{\xi'_s} \sum_{\xi'_t} |a(\{\xi_j\})|^2 \chi(\text{dist}^{(m)}(m_s, m'_s) < \ell^\delta/2) \\
& \quad \times \chi(\text{dist}^{(m)}(m_t, m'_t) < \ell^\delta/2) \chi(1 \leq m_s \leq q\ell) \\
& \quad \times \int dx_1 dy_1 \int dx_2 dy_2 |\phi_{\xi_s}^{\mathbf{P}}(\mathbf{r}_1)|^2 |U^{(2)}(x_1 - x_2, y_1 - y_2)| |\phi_{\xi'_t}^{\mathbf{P}}(\mathbf{r}_2)|^2 \\
& \leq \pi^2 (n_{\max} + 1)^2 \frac{\ell^{4\delta}}{\ell^2} \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{t \neq s} |a^*(\{\xi_j\})|^2 \chi(1 \leq m_s \leq q\ell) \\
& \quad \times \int dx_1 dy_1 \int dx_2 dy_2 |\phi_{\xi_s}^{\mathbf{P}}(\mathbf{r}_1)|^2 |U^{(2)}(x_1 - x_2, y_1 - y_2)| |\phi_{\xi'_t}^{\mathbf{P}}(\mathbf{r}_2)|^2 \\
& \leq \pi^2 (n_{\max} + 1)^3 \mathcal{C}^{(1)}(U^{(2)}) \frac{\ell^{4\delta}}{\ell^2} \sum_{\{\xi_j\}} \sum_{s=1}^N |a^*(\{\xi_j\})|^2 \chi(1 \leq m_s \leq q\ell) \\
& \leq \pi^2 q (n_{\max} + 1)^4 \mathcal{C}^{(1)}(U^{(2)}) \frac{\ell^{4\delta}}{\ell} \tag{6.59}
\end{aligned}$$

where we have used Lemma 6.1 for getting the third inequality. Similarly we have

$$\begin{aligned}
& \frac{1}{2} \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{t \neq s} \sum_{\xi'_s} \sum_{\xi'_t} |a^*(\{\xi_j\})|^2 \\
& \quad \times \left\{ 1 - \cos \left[\frac{2\pi}{\ell} (\tilde{m}(m'_s) - \tilde{m}(m_s) + \tilde{m}(m'_t) - \tilde{m}(m_t)) \right] \right\} \\
& \quad \times \chi(\text{dist}^{(m)}(m_s, m'_s) < \ell^\delta/2) \chi(\text{dist}^{(m)}(m_t, m'_t) < \ell^\delta/2) \\
& \quad \times [1 - \chi(1 \leq m_s \leq q\ell)] [1 - \chi(1 \leq m_t \leq q\ell)] \chi(1 \leq m'_s \leq q\ell) \\
& \quad \times \int dx_1 dy_1 \int dx_2 dy_2 |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_1)|^2 |U^{(2)}(x_1 - x_2, y_1 - y_2)| |\phi_{\xi'_t}^{\mathbf{P}}(\mathbf{r}_2)|^2 \\
& \leq \pi^2 \frac{\ell^{2\delta}}{\ell^2} \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{t \neq s} \sum_{\xi'_s} \sum_{\xi'_t} |a^*(\{\xi_j\})|^2 \chi(\text{dist}^{(m)}(m_s, m'_s) < \ell^\delta/2) \\
& \quad \times \chi(\text{dist}^{(m)}(m_t, m'_t) < \ell^\delta/2) \\
& \quad \times [1 - \chi(1 \leq m_s \leq q\ell)] [1 - \chi(1 \leq m_t \leq q\ell)] \chi(1 \leq m'_s \leq q\ell) \\
& \quad \times \int dx_1 dy_1 \int dx_2 dy_2 |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_1)|^2 |U^{(2)}(x_1 - x_2, y_1 - y_2)| |\phi_{\xi'_t}^{\mathbf{P}}(\mathbf{r}_2)|^2 \\
& \leq \pi^2 (n_{\max} + 1) \frac{\ell^{3\delta}}{\ell^2} \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{t \neq s} \sum_{\xi'_s} |a^*(\{\xi_j\})|^2 \chi(\text{dist}^{(m)}(m_s, m'_s) < \ell^\delta/2) \\
& \quad \times [1 - \chi(1 \leq m_s \leq q\ell)] \chi(1 \leq m'_s \leq q\ell) \\
& \quad \times \int dx_1 dy_1 \int dx_2 dy_2 |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_1)|^2 |U^{(2)}(x_1 - x_2, y_1 - y_2)| |\phi_{\xi'_t}^{\mathbf{P}}(\mathbf{r}_2)|^2 \\
& \leq \pi^2 (n_{\max} + 1)^2 \mathcal{C}^{(1)}(U^{(2)}) \frac{\ell^{3\delta}}{\ell^2} \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{\xi'_s} |a^*(\{\xi_j\})|^2 \\
& \quad \times \chi(\text{dist}^{(m)}(m_s, m'_s) < \ell^\delta/2) [1 - \chi(1 \leq m_s \leq \ell)] \chi(1 \leq m'_s \leq q\ell) \quad (6.60)
\end{aligned}$$

Note that

$$\begin{aligned}
& \sum_{s=1}^N \sum_{\xi'_s} \chi(\text{dist}^{(m)}(m_s, m'_s) < \ell^\delta/2) [1 - \chi(1 \leq m_s \leq \ell)] \chi(1 \leq m'_s \leq q\ell) \\
& \leq \sum_{s=1}^N \sum_{\xi'_s} [\chi(-\ell^\delta/2 + 1 < m_s \leq 0) + \chi(q\ell + 1 \leq m_s < \ell^\delta/2 + q\ell)] \\
& \quad \times [\chi(1 \leq m'_s < \ell^\delta/2) + \chi(q\ell + 1 - \ell^\delta/2 < m'_s \leq q\ell)] \\
& \leq (n_{\max} + 1)^2 \ell^{2\delta} \tag{6.61}
\end{aligned}$$

Substituting this into (6.60), we get

$$\begin{aligned}
& \frac{1}{2} \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{t \neq s} \sum_{\xi'_s} \sum_{\xi'_t} |a^*(\{\xi_j\})|^2 \\
& \quad \times \left\{ 1 - \cos \left[\frac{2\pi}{\ell} (\tilde{m}(m'_s) - \tilde{m}(m_s) + \tilde{m}(m'_t) - \tilde{m}(m_t)) \right] \right\} \\
& \quad \times \chi(\text{dist}^{(m)}(m_s, m'_s) < \ell^\delta/2) \chi(\text{dist}^{(m)}(m_t, m'_t) < \ell^\delta/2) \\
& \quad \times [1 - \chi(1 \leq m_s \leq q\ell)] [1 - \chi(1 \leq m_t \leq q\ell)] \chi(1 \leq m'_s \leq q\ell) \\
& \quad \times \int dx_1 dy_1 \int dx_2 dy_2 |\phi_{\xi_s}^{\mathbf{P}}(\mathbf{r}_1)|^2 |U^{(2)}(x_1 - x_2, y_1 - y_2)| |\phi_{\xi_t}^{\mathbf{P}}(\mathbf{r}_2)|^2 \\
& \leq \pi^2 (n_{\max} + 1)^4 \mathcal{C}^{(1)}(U^{(2)}) \frac{\ell^{5\delta}}{\ell^2} \tag{6.62}
\end{aligned}$$

Since the rest of the contributions of the right-hand side of (6.58) are treated in the same way, we obtain the bound (6.54).

6.2.2. Estimate of $\Delta \tilde{E}_{\Pi, >}^{(N)}$. Our goal in this subsection is to get the following bound: For $\pi(\ell^\delta/2 - 1)/L_x > n_{\max}$ and $L_y > 32n_{\max}\ell_B$,

$$\Delta \tilde{E}_{\Pi, >}^{(N)} \leq 8(n_{\max} + 1) q\ell \varepsilon^{(1)}(\ell^\delta/2 - 1, n_{\max}, L_x, L_y) \kappa(n_{\max}, L_x, L_y) \tag{6.63}$$

where

$$\begin{aligned}
\kappa(n_{\max}, L_x, L_y) := & \left\{ \mathcal{C}^{(7)}(n_{\max}) + \mathcal{C}^{(8)}(n_{\max}) \frac{L_x}{\ell_B} + \mathcal{C}^{(9)}(n_{\max}) \frac{L_x L_y}{\ell_B^2} \right. \\
& \left. \times \exp \left[-\frac{L_y^2}{32\ell_B^2} \left(1 - \frac{32n_{\max}\ell_B}{L_y} \right)^2 \right] \right\} C^{(10)}(U^{(2)}) \tag{6.64}
\end{aligned}$$

Here the constants $\mathcal{C}^{(7)}(n_{\max})$, $\mathcal{C}^{(8)}(n_{\max})$ and $\mathcal{C}^{(9)}(n_{\max})$ depend on the energy cutoff n_{\max} only, and the constant $C^{(10)}(U^{(2)})$ depends on the potential $U^{(2)}$ only. From (6.32) and (6.64), we have

$$\lim_{\ell \rightarrow \infty} \lim_{L_y \rightarrow \infty} \Delta \tilde{E}_{\Pi, >}^{(N)} = 0 \quad (6.65)$$

for a fixed L_x .

Using the definition (4.5) of $\tilde{m}(\dots)$, we can evaluate $\Delta \tilde{E}_{\Pi, >}^{(N)}$ of (6.53) as

$$\begin{aligned} \Delta \tilde{E}_{\Pi, >}^{(N)} &\leq 2 \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{t \neq s} \sum_{\xi'_s} \sum_{\xi'_t} [|a(\{\xi_j\})|^2 + |a(\{\xi'_j\})|^2] \chi(1 \leq m_s \leq q\ell) \\ &\quad \times [1 - \chi(\text{dist}^{(m)}(m_s, m'_s) < \ell^\delta/2) \chi(\text{dist}^{(m)}(m_t, m'_t) < \ell^\delta/2)] \\ &\quad \times \left| \int dx_1 dy_1 \int dx_2 dy_2 \phi_{\xi'_s}^{\mathbf{P}^*}(\mathbf{r}_1) \phi_{\xi'_t}^{\mathbf{P}^*}(\mathbf{r}_2) \right. \\ &\quad \left. \times U^{(2)}(x_1 - x_2, y_1 - y_2) \phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_1) \phi_{\xi'_t}^{\mathbf{P}}(\mathbf{r}_2) \right| \\ &\leq 2 \sum_{i=1}^4 \Delta \tilde{E}_{\Pi, >, i}^{(N)} \end{aligned} \quad (6.66)$$

in a similar way as in (6.58) in the preceding Section 6.2.1. Here

$$\begin{aligned} \Delta \tilde{E}_{\Pi, >, 1}^{(N)} &:= \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{t \neq s} \sum_{\xi'_s} \sum_{\xi'_t} |a(\{\xi_j\})|^2 \\ &\quad \times \chi(1 \leq m_s \leq q\ell) \chi(\text{dist}^{(m)}(m_s, m'_s) \geq \ell^\delta/2) \\ &\quad \times \int dx_1 dy_1 |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_1)| |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_1)| \int dx_2 dy_2 \\ &\quad \times |U^{(2)}(x_1 - x_2, y_1 - y_2)| |\phi_{\xi'_t}^{\mathbf{P}}(\mathbf{r}_2)| |\phi_{\xi'_t}^{\mathbf{P}}(\mathbf{r}_2)| \end{aligned} \quad (6.67)$$

$$\begin{aligned} \Delta \tilde{E}_{\Pi, >, 2}^{(N)} &:= \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{t \neq s} \sum_{\xi'_s} \sum_{\xi'_t} |a(\{\xi_j\})|^2 \\ &\quad \times \chi(1 \leq m_s \leq q\ell) \chi(\text{dist}^{(m)}(m_t, m'_t) \geq \ell^\delta/2) \\ &\quad \times \int dx_1 dy_1 |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_1)| |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_1)| \int dx_2 dy_2 \\ &\quad \times |U^{(2)}(x_1 - x_2, y_1 - y_2)| |\phi_{\xi'_t}^{\mathbf{P}}(\mathbf{r}_2)| |\phi_{\xi'_t}^{\mathbf{P}}(\mathbf{r}_2)| \end{aligned} \quad (6.68)$$

$$\begin{aligned}
\Delta \tilde{E}_{\Pi, >, 3}^{(N)} &:= \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{t \neq s} \sum_{\xi'_s} \sum_{\xi'_t} \\
&\quad \times |a(\{\xi''_j\})|^2 \chi(1 \leq m_s \leq q\ell) \chi(\text{dist}^{(m)}(m_s, m'_s) \geq \ell^\delta/2) \\
&\quad \times \int dx_1 dy_1 |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_1)| |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_1)| \int dx_2 dy_2 \\
&\quad \times U^{(2)}(x_1 - x_2, y_1 - y_2) |\phi_{\xi'_t}^{\mathbf{P}}(\mathbf{r}_2)| |\phi_{\xi'_t}^{\mathbf{P}}(\mathbf{r}_2)| \quad (6.69)
\end{aligned}$$

and

$$\begin{aligned}
\Delta \tilde{E}_{\Pi, >, 4}^{(N)} &:= \sum_{\{\xi_j\}} \sum_{s=1}^N \sum_{t \neq s} \sum_{\xi'_s} \sum_{\xi'_t} |a(\{\xi''_j\})|^2 \\
&\quad \times \chi(1 \leq m_s \leq q\ell) \chi(\text{dist}^{(m)}(m_t, m'_t) \geq \ell^\delta/2) \\
&\quad \times \int dx_1 dy_1 |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_1)| |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_1)| \int dx_2 dy_2 \\
&\quad \times |U^{(2)}(x_1 - x_2, y_1 - y_2)| |\phi_{\xi'_t}^{\mathbf{P}}(\mathbf{r}_2)| |\phi_{\xi'_t}^{\mathbf{P}}(\mathbf{r}_2)| \quad (6.70)
\end{aligned}$$

and we have used

$$\begin{aligned}
1 - \chi(\text{dist}^{(m)}(m_s, m'_s) < \ell^\delta/2) \chi(\text{dist}^{(m)}(m_t, m'_t) < \ell^\delta/2) \\
\leq \chi(\text{dist}^{(m)}(m_s, m'_s) \geq \ell^\delta/2) + \chi(\text{dist}^{(m)}(m_t, m'_t) \geq \ell^\delta/2) \quad (6.71)
\end{aligned}$$

Consider first $\Delta \tilde{E}_{\Pi, >, 1}^{(N)}$. It can be written as

$$\begin{aligned}
\Delta \tilde{E}_{\Pi, >, 1}^{(N)} &= \sum_{\{\xi_j\}} |a(\{\xi_j\})|^2 \sum_{s=1}^N \chi(1 \leq m_s \leq q\ell) \\
&\quad \times \sum_{\xi'_s} \chi(\text{dist}^{(m)}(m_s, m'_s) \geq \ell^\delta/2) \int dx_1 dy_1 |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_1)| |\phi_{\xi'_s}^{\mathbf{P}}(\mathbf{r}_1)| \\
&\quad \times \sum_{t \neq s} \sum_{\xi'_t} \int dx_2 dy_2 |U^{(2)}(x_2 - x_1, y_2 - y_1)| |\phi_{\xi'_t}^{\mathbf{P}}(\mathbf{r}_2)| |\phi_{\xi'_t}^{\mathbf{P}}(\mathbf{r}_2)| \quad (6.72)
\end{aligned}$$

In order to evaluate the last two sums, we use the following lemma:

Lemma 6.4. Let $L_y > 32n_{\max}\ell_B$. Then

$$\sum_{\xi} \sum_{\xi'} \int_S dx dy |U^{(2)}(x - x', y - y')| |\phi_{\xi}^{\mathbf{P}}(x, y)| |\phi_{\xi'}^{\mathbf{P}}(x, y)| \leq \kappa(n_{\max}, L_x, L_y) \quad (6.73)$$

for all $x', y' \in \mathbf{R}$. Here κ is given by (6.64).

The proof is given in Appendix E. Using this Lemma 6.4, (6.72) and Lemma 6.3, we get

$$\begin{aligned} \Delta \tilde{E}_{\text{II}, >, 1}^{(N)} &\leq \sum_{\{\xi_j\}} |a(\{\xi_j\})|^2 \sum_{s=1}^N \chi(1 \leq m_s \leq q\ell) \\ &\quad \times \varepsilon^{(1)}(\ell^\delta/2 - 1, n_{\max}, L_x, L_y) \kappa(n_{\max}, L_x, L_y) \\ &\leq (n_{\max} + 1) q\ell \varepsilon^{(1)}(\ell^\delta/2 - 1, n_{\max}, L_x, L_y) \kappa(n_{\max}, L_x, L_y) \end{aligned} \quad (6.74)$$

for ℓ satisfying $\pi(\ell^\delta/2 - 1) \ell_B/L_x > n_{\max}$.

Next consider $\Delta \tilde{E}_{\text{II}, >, 3}^{(N)}$. One can easily get

$$\begin{aligned} \Delta \tilde{E}_{\text{II}, >, 3}^{(N)} &= \sum_{\{\xi_j''\}} \sum_{s=1}^N \sum_{t \neq s} \sum_{\xi_s} \sum_{\xi_t} |a(\{\xi_j''\})|^2 \\ &\quad \times \chi(1 \leq m_s \leq q\ell) \chi(\text{dist}^{(m)}(m_s, m'_s) \geq \ell^\delta/2) \\ &\quad \times \int dx_1 dy_1 |\phi_{\xi_s}^{\text{P}}(\mathbf{r}_1)| |\phi_{\xi'_s}^{\text{P}}(\mathbf{r}_1)| \\ &\quad \times \int dx_2 dy_2 |U^{(2)}(x_1 - x_2, y_1 - y_2)| |\phi_{\xi_t}^{\text{P}}(\mathbf{r}_2)| |\phi_{\xi'_t}^{\text{P}}(\mathbf{r}_2)| \\ &\leq \sum_{\{\xi_j''\}} |a(\{\xi_j''\})|^2 \sum_{\xi} \chi(1 \leq m \leq q\ell) \\ &\quad \times \sum_{s=1}^N \chi(\text{dist}^{(m)}(m, m'_s) \geq \ell^\delta/2) \int dx_1 dy_1 |\phi_{\xi}^{\text{P}}(\mathbf{r}_1)| |\phi_{\xi'_s}^{\text{P}}(\mathbf{r}_1)| \\ &\quad \times \sum_{t=1}^N \sum_{\xi_t} \int dx_2 dy_2 |U^{(2)}(x_2 - x_1, y_2 - y_1)| |\phi_{\xi_t}^{\text{P}}(\mathbf{r}_2)| |\phi_{\xi'_t}^{\text{P}}(\mathbf{r}_2)| \end{aligned} \quad (6.75)$$

Here the sum about ξ is over all the states $\xi = (n, k) = (n, 2\pi m/L_x)$ with the Landau index $n \leq n_{\max}$. Therefore $\Delta \tilde{E}_{\text{II}, >, 3}^{(N)}$ can be treated in the same way as $\Delta \tilde{E}_{\text{II}, >, 1}^{(N)}$. Moreover the rest $\Delta \tilde{E}_{\text{II}, >, 2}^{(N)}$ and $\Delta \tilde{E}_{\text{II}, >, 4}^{(N)}$ also can be treated in the same way. In a consequence, we obtain the desired bound (6.63) from (6.66) and (6.74).

APPENDIX A. PROOF OF THEOREM 2.1

Following Matsui, we sketch the proof of Theorem 2.1. For the detail, see ref. 26.

Let Λ be a one-dimensional finite lattice, i.e., $\Lambda \subset \mathbf{Z}$. For simplicity, we assume $|\Lambda| = qL$ with positive integers q, L . Then there exists a self-adjoint operator $\tilde{h}_0^{(q)}$, i.e., a local Hamiltonian, such that the Hamiltonian (2.28) can be written as

$$\tilde{H}_\Lambda(n_{\max}) = \sum_{m=0}^{L-1} \tilde{h}_{qm}^{(q)} \quad (\text{A.1})$$

in terms of the translate $\tilde{h}_x^{(q)} := \tau_x^{(y)}(\tilde{h}_0^{(q)})$ of the local Hamiltonian, with the periodic boundary conditions. We introduce a number operator of electron on q lattice sites as

$$\tilde{n}_x^{(q)} := \sum_{n=0}^{n_{\max}} \sum_{m=1}^q \tilde{n}_{n, x+m} \quad (\text{A.2})$$

Here $\tilde{n}_{n, m}$ is given by (2.38). Thereby the Hamiltonian (2.37) with a chemical potential μ is written as

$$\tilde{H}_{\Lambda, \mu}(n_{\max}) = \sum_{m=0}^{L-1} [\tilde{h}_{qm}^{(q)} - \mu \tilde{n}_{qm}^{(q)}] \quad (\text{A.3})$$

To begin with, we recall the following two theorems:

Theorem A.1. Let ω be a translationally invariant state with a period q . Then the following two conditions for the grand-canonical ensemble with a chemical potential μ are equivalent:

- ω is a ground state for \mathcal{A}_{loc} .
- ω minimizes the local energy in the sense that

$$\omega(\tilde{h}_x^{(q)} - \mu \tilde{n}_x^{(q)}) = \inf \psi(\tilde{h}_x^{(q)} - \mu \tilde{n}_x^{(q)}) \quad (\text{A.4})$$

where the inf is taken over all the translationally invariant states.

Theorem A.2. Let ω be a translationally invariant state with a period q and with the local density $\omega(\tilde{n}_x^{(q)}) = \rho$. Then the following two conditions for the canonical ensemble are equivalent:

- ω is a ground state for $\mathcal{A}_{\text{loc}}^{U(1)}$.
- ω minimizes the local energy in the sense that

$$\omega(\tilde{h}_x^{(q)}) = \inf \psi(\tilde{h}_x^{(q)}) \quad (\text{A.5})$$

where the inf is taken over all the translationally invariant states with the local density $\psi(\tilde{n}_x^{(q)}) = \rho$.

In order to show these statements, one has only to estimate energy effects due to boundary conditions, by relying on the Bratteli–Kishimoto–Robinson theorem.⁽³¹⁾ See, for example, ref. 26. See also refs. 22 and 27.

Lemma A.3. Let ω be a translationally invariant ground state with an electron density $\omega(\tilde{n}_x^{(q)}) = \rho$ for $\mathcal{A}_{\text{loc}}^{U(1)}$. Suppose that, for \mathcal{A}_{loc} , there exists a translationally invariant ground state η with the same electron density $\eta(\tilde{n}_x^{(q)}) = \rho$ and with a chemical potential μ . Then the gauge invariant extension $\tilde{\omega}$ of ω to \mathcal{A}_{loc} is a ground state for \mathcal{A}_{loc} , with the chemical potential μ .

Proof. We note that, for $a \in \mathcal{A}_{\text{loc}}^{U(1)}$,

$$\lim_{A \uparrow \mathbf{Z}} \eta(a^*[\tilde{H}_{A,\mu}, a]) = \lim_{A \uparrow \mathbf{Z}} \eta(a^*[\tilde{H}_A, a]) \geq 0 \quad (\text{A.6})$$

because the operator a commutes with the total number operator of electron. This implies that η is a translationally invariant ground state for $\mathcal{A}_{\text{loc}}^{U(1)}$. Therefore

$$\eta(\tilde{h}_x^{(q)}) = \omega(\tilde{h}_x^{(q)}) \quad (\text{A.7})$$

owing to Theorem A.2. Since η and ω have the same electron density ρ , one has

$$\eta(\tilde{h}_x^{(q)} - \mu\tilde{n}_x^{(q)}) = \omega(\tilde{h}_x^{(q)} - \mu\tilde{n}_x^{(q)}) \quad (\text{A.8})$$

This implies that $\tilde{\omega}$ is a translationally invariant ground state for \mathcal{A}_{loc} , from Theorem A.1. ■

Proof of Theorem 2.1. By this Lemma, it is sufficient to show that, for any given electron density ρ , there exists a chemical potential μ such that a ground state η for \mathcal{A}_{loc} with μ has the density ρ .

Let $\tilde{\Phi}_{A,\mu}$ be a ground state of the Hamiltonian $\tilde{H}_{A,\mu}$ with a chemical potential μ such that the corresponding expectation $\eta_{A,\mu}(\dots) = \langle \tilde{\Phi}_{A,\mu}, (\dots) \tilde{\Phi}_{A,\mu} \rangle$ is translationally invariant. Then the corresponding infinite-volume state $\eta_\mu = w^* - \lim_{A \uparrow \mathbf{Z}} \eta_{A,\mu}$ is a translationally invariant ground state with the chemical potential μ for \mathcal{A}_{loc} . From Theorem A.1, the following two inequalities are valid:

$$\eta_\mu(\tilde{h}_x^{(q)}) - \eta_{\mu'}(\tilde{h}_x^{(q)}) \leq \mu[\eta_\mu(\tilde{n}_x^{(q)}) - \eta_{\mu'}(\tilde{n}_x^{(q)})] \quad (\text{A.9})$$

and

$$\eta_{\mu'}(\tilde{n}_x^{(q)}) - \eta_{\mu}(\tilde{n}_x^{(q)}) \leq \mu' [\eta_{\mu'}(\tilde{n}_x^{(q)}) - \eta_{\mu}(\tilde{n}_x^{(q)})] \quad (\text{A.10})$$

for the infinite-volume ground states η_{μ} and $\eta_{\mu'}$ with the chemical potentials μ and μ' , respectively. By adding both sides, one has

$$0 \leq (\mu - \mu') [\eta_{\mu}(\tilde{n}_x^{(q)}) - \eta_{\mu'}(\tilde{n}_x^{(q)})] \quad (\text{A.11})$$

This implies that the electron density $\rho_{\mu} := \eta_{\mu}(\tilde{n}_x^{(q)})$ is a non-decreasing function of the chemical potential μ . As is well known, all the discontinuous points of a non-decreasing function is at most countable. Assume that μ_0 is such a discontinuous point. Namely,

$$\eta_{\mu_0}^- = \lim_{\mu \uparrow \mu_0} \eta_{\mu}, \quad \eta_{\mu_0}^+ = \lim_{\mu \downarrow \mu_0} \eta_{\mu} \quad (\text{A.12})$$

with

$$\rho_{\mu_0}^- := \eta_{\mu_0}^-(\tilde{n}_x^{(q)}) \neq \eta_{\mu_0}^+(\tilde{n}_x^{(q)}) =: \rho_{\mu_0}^+ \quad (\text{A.13})$$

Consider the convex combination $\eta_{\mu_0}^{\lambda} := \lambda \eta_{\mu_0}^- + (1 - \lambda) \eta_{\mu_0}^+$ with $\lambda \in [0, 1]$. Clearly the state $\eta_{\mu_0}^{\lambda}$ is a translationally invariant ground state for \mathcal{A}_{loc} and for any $\lambda \in [0, 1]$. Further $\eta_{\mu_0}^{\lambda}$ has the electron density $\lambda \rho_{\mu_0}^- + (1 - \lambda) \rho_{\mu_0}^+$. This continuously interpolates between the two densities $\rho_{\mu_0}^-, \rho_{\mu_0}^+$. ■

APPENDIX B. PROOF OF LEMMA 6.1

Consider a density function given by

$$\rho_n(x, y) := \sum_k |\phi_{n,k}^{\text{P}}(x, y)|^2 \quad (\text{B.1})$$

From the expression (3.17) of $\phi_{n,k}^{\text{P}}$, this function ρ_n is periodic in both x and y directions as

$$\rho_n(x, y) = \rho_n(x + \Delta x, y) = \rho_n(x, y + \Delta y) \quad (\text{B.2})$$

Here Δx and Δy are given by (3.13). Owing to this periodicity, the integral of ρ_n on the unit cell $\Delta_{\ell, m}$ becomes

$$\int_{\Delta_{\ell, m}} dx dy \rho_n(x, y) = \frac{1}{M} \quad (\text{B.3})$$

where

$$\Delta_m := [x_\ell, x_{\ell+1}] \times [y_m, y_{m+1}] \quad (\text{B.4})$$

with

$$x_\ell = -\frac{L_x}{2} + (\ell - 1) \Delta x \quad \text{for } \ell = 1, 2, \dots, M \quad (\text{B.5})$$

and

$$y_m = -\frac{L_y}{2} + (m - 1) \Delta y \quad \text{for } m = 1, 2, \dots, M \quad (\text{B.6})$$

Clearly we have

$$\begin{aligned} \sum_k \int_S dx dy |U^{(2)}(x - x', y - y')| |\phi_{n,k}^P(x, y)|^2 \\ = \int_S dx dy |U^{(2)}(x - x', y - y')| \rho_n(x, y) \end{aligned} \quad (\text{B.7})$$

Combining this, the periodicity (B.2) of ρ_n and the periodicity (2.7) of $U^{(2)}$, we can assume

$$|x'| \leq \frac{\Delta x}{2}, \quad |y'| \leq \frac{\Delta y}{2} \quad (\text{B.8})$$

for showing the statement of Lemma 6.1.

Since the function $U^{(2)}$ is continuous by the assumption, we have

$$\begin{aligned} \int_{\Delta_{\ell,m}} dx dy |U^{(2)}(x - x', y - y')| \rho_n(x, y) \\ = |U^{(2)}(\xi_{\ell,m} - x', \eta_{\ell,m} - y')| \int_{\Delta_{\ell,m}} dx dy \rho_n(x, y) \\ = \frac{1}{M} |U^{(2)}(\xi_{\ell,m} - x', \eta_{\ell,m} - y')| \end{aligned} \quad (\text{B.9})$$

where $(\xi_{\ell,m}, \eta_{\ell,m})$ is a point in the cell $\Delta_{\ell,m}$, and we have used (B.3). Thereby

$$\begin{aligned}
& \int_S dx dy |U^{(2)}(x-x', y-y')| \rho_n(x, y) \\
&= \frac{eB}{h} \sum_{\ell,m} |U^{(2)}(\xi_{\ell,m}-x', \eta_{\ell,m}-y')| \Delta x \Delta y \\
&= \frac{eB}{h} \sum_{r_{\ell,m} \leq R} |U^{(2)}(\xi_{\ell,m}-x', \eta_{\ell,m}-y')| \Delta x \Delta y \\
&\quad + \frac{eB}{h} \sum_{r_{\ell,m} > R} |U^{(2)}(\xi_{\ell,m}-x', \eta_{\ell,m}-y')| \Delta x \Delta y \quad (\text{B.10})
\end{aligned}$$

where $r_{\ell,m} = \sqrt{\xi_{\ell,m}^2 + \eta_{\ell,m}^2}$. The first sum in the right-hand side of the second equality converges to

$$\int_{x^2+y^2 \leq R^2} dx dy |U^{(2)}(x, y)| \quad (\text{B.11})$$

as $L_x, L_y \rightarrow +\infty$ for a fixed R because $|U^{(2)}|$ is uniformly continuous. The second sum becomes small for a large R from the assumption (2.8) of $U^{(2)}$ on the decay for a large distance. From these observations, we get

$$\sum_n \int_S dx dy |U^{(2)}(x-x', y-y')| \rho_n(x, y) \leq \mathcal{C}^{(1)}(U^{(2)})(n_{\max} + 1) \quad (\text{B.12})$$

The finite constant $\mathcal{C}^{(1)}(U^{(2)})$ depends only on $U^{(2)}$. Thus the statement of Lemma 6.1 has been proved.

APPENDIX C. PROOF OF LEMMA 6.3

In order to prove Lemma 6.3, we use the following estimate for the integral in the left-hand side of (6.41):

Lemma C.1. Let $L_y > 32n_{\max}\ell_B$, and let $n, n' \leq n_{\max}$. Then the following bound is valid:

$$\int_{-L_x/2}^{L_x/2} dx \int_{-L_y/2}^{L_y/2} dy |\phi_{n',k'}^{\text{P}}(\mathbf{r})| |\phi_{n,k}^{\text{P}}(\mathbf{r})| \leq \varepsilon^{(2)}(\text{dist}^{(m)}(m, m'), n_{\max}, L_y) \quad (\text{C.1})$$

where $k = 2\pi m/L_x$, $k' = 2\pi m'/L_x$, and

$$\begin{aligned} \varepsilon^{(2)}(\Delta\ell, n_{\max}, L_y) &:= \mathcal{C}^{(4)}(n_{\max}) \exp \left[-\left(\frac{\pi\ell_B}{L_x} \Delta\ell - n_{\max} \right)^2 \right] \\ &+ \mathcal{C}^{(5)}(n_{\max}) \exp \left[-\frac{L_y^2}{32\ell_B^2} \left(1 - \frac{32n_{\max}\ell_B}{L_y} \right)^2 \right] \end{aligned} \quad (\text{C.2})$$

Here the constants $\mathcal{C}^{(4)}(n_{\max})$ and $\mathcal{C}^{(5)}(n_{\max})$ depend on the energy cutoff n_{\max} only.

The proof is given in the next Appendix D. By using the bound (C.1), we have

$$\begin{aligned} \sum_{\xi'} \chi(\text{dist}^{(m)}(m, m') \geq \Delta\ell) \int_S dx dy |\phi_{\xi'}^P(\mathbf{r})| |\phi_{\xi}^P(\mathbf{r})| \\ \leq 2(n_{\max} + 1) \mathcal{C}^{(4)}(n_{\max}) \sum_{\ell=\Delta\ell}^{\infty} \exp \left[-\left(\frac{\pi\ell_B}{L_x} \ell - n_{\max} \right)^2 \right] \\ + (n_{\max} + 1) \frac{L_x L_y}{2\pi\ell_B^2} \mathcal{C}^{(5)}(n_{\max}) \exp \left[-\frac{L_y^2}{32\ell_B^2} \left(1 - \frac{32n_{\max}\ell_B}{L_y} \right)^2 \right] \end{aligned} \quad (\text{C.3})$$

The sum in the right-hand side is evaluated as

$$\begin{aligned} \sum_{\ell=\Delta\ell}^{\infty} \exp \left[-\left(\frac{\pi\ell_B}{L_x} \ell - n_{\max} \right)^2 \right] \\ \leq \int_{\Delta\ell-1}^{\infty} d\ell \exp \left[-\left(\frac{\pi\ell_B}{L_x} \ell - n_{\max} \right)^2 \right] \\ \leq \int_0^{\infty} d\ell \exp \left[-\left\{ \frac{\pi\ell_B}{L_x} (\ell + \Delta\ell - 1) - n_{\max} \right\}^2 \right] \\ \leq \int_0^{\infty} d\ell \exp \left[-\frac{\pi^2\ell_B^2}{L_x^2} \ell^2 \right] \exp \left[-\left\{ \frac{\pi\ell_B}{L_x} (\Delta\ell - 1) - n_{\max} \right\}^2 \right] \\ = \frac{1}{2\sqrt{\pi}} \frac{L_x}{\ell_B} \exp \left[-\left\{ \frac{\pi\ell_B}{L_x} (\Delta\ell - 1) - n_{\max} \right\}^2 \right] \end{aligned} \quad (\text{C.4})$$

Here we have used the assumption $\pi(\Delta\ell - 1)\ell_B/L_x > n_{\max}$ of Lemma 6.3. Substituting this into (C.3), we obtain the desired bound (6.41) with (6.32).

APPENDIX D. PROOF OF LEMMA C.1

Throughout the present Appendix we assume $L_y > 32n_{\max}\ell_B$ which is the assumption of Lemma C.1.

Using the expression (3.17) of $\phi_{n,k}^P$, we evaluate the integral of the left-hand side of (C.1) as

$$\begin{aligned}
 & \int_{-L_x/2}^{L_x/2} dx \int_{-L_y/2}^{L_y/2} dy |\phi_{n',k'}^P(\mathbf{r})| |\phi_{n,k}^P(\mathbf{r})| \\
 & \leq \sum_{\ell, \ell'} \int_{-L_y/2}^{L_y/2} dy |v_{n'}(y - y_{k'} - \ell' L_y)| |v_n(y - y_k - \ell L_y)| \\
 & \leq \sum_{\ell, \ell'} \int_{-L_y/2}^{L_y/2} d\tilde{y} \left| v_{n'} \left(\tilde{y} + \frac{y_k - y_{k'}}{2} - \ell' L_y \right) \right| \left| v_n \left(\tilde{y} - \frac{y_k - y_{k'}}{2} - \ell L_y \right) \right|
 \end{aligned} \tag{D.1}$$

where we have used the periodicity of the integrand, and have changed the variable as

$$\tilde{y} = y - \frac{y_k + y_{k'}}{2} \tag{D.2}$$

for getting the second inequality. From the right-hand side of the first inequality, we can assume $|y_k - y_{k'}| \leq L_y/2$ without loss of generality.

Lemma D.1. Let $|y| \leq 3L_y/4$. Then

$$\sum_{\ell=1}^{\infty} |v_n(y \pm \ell L_y)| \leq \mathcal{C}^{(6)}(n_{\max}) \exp \left[-\frac{L_y^2}{32\ell^2 B} \left(1 - \frac{32n_{\max}\ell_B}{L_y} \right)^2 \right] \tag{D.3}$$

where the constant is given by

$$\mathcal{C}^{(6)}(n_{\max}) := \left(1 + \frac{\sqrt{2\pi}}{16n_{\max}} \right) \max_{n \leq n_{\max}} \{c_n N_n \exp[32\beta_n^2]\} \tag{D.4}$$

Proof. Using the bound (6.38) for the Hermite polynomial H_n and the assumption $|y| \leq 3L_y/4$, we have

$$\begin{aligned}
\sum_{\ell=1}^{\infty} |v_n(y \pm \ell L_y)| &\leq c_n N_n \sum_{\ell=1}^{\infty} \exp[\beta_n(\ell + 3/4) L_y/\ell_B] \exp\left[-\frac{(\ell - 3/4)^2 L_y^2}{2\ell_B^2}\right] \\
&\leq c_n N_n \sum_{\ell=1}^{\infty} \exp\left[\frac{2\beta_n \ell L_y}{\ell_B}\right] \exp\left[-\frac{\ell^2 L_y^2}{32\ell_B^2}\right] \\
&= c_n N_n \exp[32\beta_n^2] \sum_{\ell=1}^{\infty} \exp\left[-\frac{L_y^2}{32\ell_B^2} \left(\ell - \frac{32\beta_n \ell_B}{L_y}\right)^2\right]
\end{aligned} \tag{D.5}$$

Here we have used

$$\ell - \frac{3}{4} \geq \frac{\ell}{4}, \quad \text{and} \quad \ell + \frac{3}{4} \leq 2\ell \tag{D.6}$$

for getting the second inequality. The sum in the last line of (D.5) is evaluated as

$$\begin{aligned}
&\sum_{\ell=1}^{\infty} \exp\left[-\frac{L_y^2}{32\ell_B^2} \left(\ell - \frac{32\beta_n \ell_B}{L_y}\right)^2\right] \\
&\leq \exp\left[-\frac{L_y^2}{32\ell_B^2} \left(1 - \frac{32\beta_n \ell_B}{L_y}\right)^2\right] \\
&\quad + \int_0^{\infty} dy \exp\left[-\frac{L_y^2}{32\ell_B^2} \left(y + 1 - \frac{32\beta_n \ell_B}{L_y}\right)^2\right] \\
&\leq \exp\left[-\frac{L_y^2}{32\ell_B^2} \left(1 - \frac{32\beta_n \ell_B}{L_y}\right)^2\right] \\
&\quad + \frac{1}{2} \sqrt{\frac{32\ell_B^2 \pi}{L_y^2}} \exp\left[-\frac{L_y^2}{32\ell_B^2} \left(1 - \frac{32\beta_n \ell_B}{L_y}\right)^2\right] \\
&= \left(1 + 2\sqrt{2\pi} \frac{\ell_B}{L_y}\right) \exp\left[-\frac{L_y^2}{32\ell_B^2} \left(1 - \frac{32\beta_n \ell_B}{L_y}\right)^2\right] \\
&\leq \left(1 + \frac{\sqrt{2\pi}}{16n_{\max}}\right) \exp\left[-\frac{L_y^2}{32\ell_B^2} \left(1 - \frac{32\beta_n \ell_B}{L_y}\right)^2\right]
\end{aligned} \tag{D.7}$$

by using the assumption $L_y > 32n_{\max} \ell_B \geq 32\beta_n \ell_B$. ■

Using the bound (6.38) for the Hermite polynomial H_n , we have

$$\begin{aligned}
 \int_{-\infty}^{+\infty} dy |v_n(y)| &\leq c_n N_n \int_{-\infty}^{+\infty} dy \exp\left[\frac{\beta_n |y|}{\ell_B}\right] \exp\left[-\frac{y^2}{2\ell_B^2}\right] \\
 &\leq 2c_n N_n \int_0^{+\infty} dy \exp\left[-\frac{1}{2\ell_B^2}(y - \beta_n \ell_B)^2\right] \exp\left[\frac{1}{2}\beta_n^2\right] \\
 &\leq 2c_n N_n \exp\left[\frac{1}{2}\beta_n^2\right] \int_{-\infty}^{+\infty} dy \exp\left[-\frac{y^2}{2\ell_B^2}\right] \\
 &= 2\sqrt{2\pi} c_n N_n \exp\left[\frac{1}{2}\beta_n^2\right] \ell_B
 \end{aligned} \tag{D.8}$$

From this inequality and Lemma D.1, we have

$$\begin{aligned}
 &\sum_{\ell, \ell'} \int_{-L_y/2}^{L_y/2} d\tilde{y} \left| v_{n'}\left(\tilde{y} + \frac{y_k - y_{k'}}{2} - \ell' L_y\right) \right| \left| v_n\left(\tilde{y} - \frac{y_k - y_{k'}}{2} - \ell L_y\right) \right| \\
 &\leq \int_{-L_y/2}^{L_y/2} d\tilde{y} \left| v_{n'}\left(\tilde{y} + \frac{y_k - y_{k'}}{2}\right) \right| \left| v_n\left(\tilde{y} - \frac{y_k - y_{k'}}{2}\right) \right| \\
 &\quad + \sum_{\ell} \sum_{\ell' \neq 0} \int_{-L_y/2}^{L_y/2} d\tilde{y} \left| v_{n'}\left(\tilde{y} + \frac{y_k - y_{k'}}{2} - \ell' L_y\right) \right| \\
 &\quad \times \left| v_n\left(\tilde{y} - \frac{y_k - y_{k'}}{2} - \ell L_y\right) \right| \\
 &\quad + \sum_{\ell'} \sum_{\ell \neq 0} \int_{-L_y/2}^{L_y/2} d\tilde{y} \left| v_{n'}\left(\tilde{y} + \frac{y_k - y_{k'}}{2} - \ell' L_y\right) \right| \\
 &\quad \times \left| v_n\left(\tilde{y} - \frac{y_k - y_{k'}}{2} - \ell L_y\right) \right| \\
 &\leq \int_{-L_y/2}^{L_y/2} d\tilde{y} \left| v_{n'}\left(\tilde{y} + \frac{y_k - y_{k'}}{2}\right) \right| \left| v_n\left(\tilde{y} - \frac{y_k - y_{k'}}{2}\right) \right| \\
 &\quad + \mathcal{C}^{(5)}(n_{\max}) \exp\left[-\frac{L_y^2}{32\ell_B^2} \left(1 - \frac{32n_{\max}\ell_B}{L_y}\right)^2\right]
 \end{aligned} \tag{D.9}$$

where

$$\mathcal{C}^{(5)}(n_{\max}) := 8\sqrt{2\pi} \ell_B \mathcal{C}^{(6)}(n_{\max}) \max_{n \leq n_{\max}} \{c_n N_n \exp[\beta_n^2/2]\} \tag{D.10}$$

Using the bound (6.38) for the Hermite polynomial, the rest of the integral in (D.9) can be evaluated as

$$\begin{aligned}
& \int_{-L_y/2}^{L_y/2} d\tilde{y} |v_{n'}(y + \delta y)| |v_n(y - \delta y)| \\
& \leq c_n c_{n'} N_n N_{n'} \int_{-L_y/2}^{L_y/2} d\tilde{y} \exp \left[n_{\max} \left(\frac{|\tilde{y} + \delta y|}{\ell_B} + \frac{|\tilde{y} - \delta y|}{\ell_B} \right) \right] \\
& \quad \times \exp \left[-\frac{\tilde{y}^2}{\ell_B^2} - \frac{\delta y^2}{\ell_B^2} \right] \\
& \leq c_n c_{n'} N_n N_{n'} \exp \left[-\frac{\delta y^2}{\ell_B^2} + 2n_{\max} \frac{|\delta y|}{\ell_B} \right] \int_{-L_y/2}^{L_y/2} d\tilde{y} \\
& \quad \times \exp \left[-\frac{\tilde{y}^2}{\ell_B^2} + 2n_{\max} \frac{|\tilde{y}|}{\ell_B} \right] \\
& \leq 2 \sqrt{\pi} \ell_B c_n c_{n'} N_n N_{n'} \exp[2n_{\max}^2] \exp \left[-\left(\frac{|\delta y|}{\ell_B} - n_{\max} \right)^2 \right] \quad (D.11)
\end{aligned}$$

for $\delta y \in \mathbf{R}$. Combining (D.1), (D.9) and (D.11), we obtain the desired bound (C.1).

APPENDIX E. PROOF OF LEMMA 6.4

Throughout the present Appendix, we assume $L_y > 32n_{\max}\ell_B$ which is the assumption of Lemma 6.4.

Note that the right hand side of (6.73) is written as

$$\begin{aligned}
& \sum_{\xi} \sum_{\xi'} \int_S dx dy |U^{(2)}(x - x', y - y')| |\phi_{\xi}^{\mathbf{P}}(x, y)| |\phi_{\xi'}^{\mathbf{P}}(x, y)| \\
& = \sum_{n=0}^{n_{\max}} \sum_{n'=0}^{n_{\max}} \sum_{\Delta\ell} \sum_k \int_S dx dy |U^{(2)}(x - x', y - y')| |\phi_{\xi}^{\mathbf{P}}(x, y)| |\phi_{\xi'}^{\mathbf{P}}(x, y)| \\
& \leq (n_{\max} + 1)^2 \max_{n, n'} \left\{ \sum_{\Delta\ell} \sum_k \int_S dx dy |U^{(2)}(x - x', y - y')| \right. \\
& \quad \left. \times |\phi_{\xi}^{\mathbf{P}}(x, y)| |\phi_{\xi'}^{\mathbf{P}}(x, y)| \right\} \quad (E.1)
\end{aligned}$$

where $k' = k + 2\pi\Delta\ell/L_x$. In order to estimate the right-hand side of the inequality, we use the following lemma which is an extension of Lemma 6.1:

Lemma E.1. Let n, n' be indices of the Landau levels, and let $\Delta\ell$ be a positive integer. Then

$$\begin{aligned} \sum_k \int_S dx dy |U^{(2)}(x-x', y-y')| |\phi_{n,k}^P(\mathbf{r})| |\phi_{n',k'}^P(\mathbf{r})| \\ \leq \varepsilon^{(2)}(\Delta\ell, n_{\max}, L_x, L_y) \mathcal{C}^{(10)}(U^{(2)}) \end{aligned} \quad (\text{E.2})$$

for $x', y' \in \mathbf{R}$, where $k' = k + 2\pi\Delta\ell/L_x$, $\varepsilon^{(2)}(\Delta\ell, n_{\max}, L_x, L_y)$ is given by (C.2), and $\mathcal{C}^{(10)}(U^{(2)})$ is a positive constant which depends on the potential $U^{(2)}$ only.

Proof. Consider a density function

$$\rho_{n,n'}(\mathbf{r}; \Delta\ell) = \sum_k |\phi_{n,k}^P(\mathbf{r})| |\phi_{n',k'}^P(\mathbf{r})| \quad (\text{E.3})$$

Then we have

$$\begin{aligned} \sum_k \int_S dx dy |U^{(2)}(x-x', y-y')| |\phi_{n,k}^P(\mathbf{r})| |\phi_{n',k'}^P(\mathbf{r})| \\ = \int_S dx dy |U^{(2)}(x-x', y-y')| \rho_{n,n'}(x, y; \Delta\ell) \end{aligned} \quad (\text{E.4})$$

From the expression (3.17) of $\Phi_{n,k}^P$, one can notice that this density function is periodic in both x and y directions as

$$\rho_{n,n'}(x + \Delta x, y; \Delta\ell) = \rho_{n,n'}(x, y + \Delta y; \Delta\ell) = \rho_{n,n'}(x, y; \Delta\ell) \quad (\text{E.5})$$

Owing to this property and the periodicity (2.7) of $U^{(2)}$, we can assume

$$|x'| \leq \frac{1}{2}\Delta x, \quad |y'| \leq \frac{1}{2}\Delta y \quad (\text{E.6})$$

for evaluating (E.4). Further we have

$$\begin{aligned} \int_S dx dy \rho_{n,n'}(x, y; \Delta\ell) &= M^2 \int_{\Delta\ell, m} dx dy \rho_{n,n'}(x, y; \Delta\ell) \\ &\leq M \max_k \int_S dx dy |\phi_{n,k}^P(\mathbf{r})| |\phi_{n',k'}^P(\mathbf{r})| \end{aligned} \quad (\text{E.7})$$

where $\Delta\ell, m$ are the unit cells given by

$$\Delta\ell, m := [x_\ell, x_{\ell+1}] \times [y_m, y_{m+1}] \quad (\text{E.8})$$

with

$$x_\ell = -\frac{L_x}{2} + (\ell - 1) \Delta x \quad \text{for } \ell = 1, 2, \dots, M \quad (\text{E.9})$$

$$y_m = -\frac{L_y}{2} + (m - 1) \Delta y \quad \text{for } m = 1, 2, \dots, M \quad (\text{E.10})$$

Since the right-hand side of the inequality in (E.7) is evaluated by using Lemma C.1, we get

$$\int_{\mathcal{A}_{\ell, m}} dx dy \rho_{n, n'}(x, y; \Delta \ell) \leq \frac{1}{M} \varepsilon^{(2)}(\Delta \ell, n_{\max}, L_x, L_y) \quad (\text{E.11})$$

From this inequality and the assumption that $U^{(2)}$ is continuous, we have

$$\begin{aligned} & \int_{\mathcal{A}_{\ell, m}} dx dy |U^{(2)}(x - x', y - y')| \rho_{n, n'}(x, y; \Delta \ell) \\ &= |U^{(2)}(\xi_{\ell, m} - x', \eta_{\ell, m} - y')| \int_{\mathcal{A}_{\ell, m}} dx dy \rho_{n, n'}(x, y; \Delta \ell) \\ &\leq |U^{(2)}(\xi_{\ell, m} - x', \eta_{\ell, m} - y')| \frac{1}{M} \varepsilon^{(2)}(\Delta \ell, n_{\max}, L_x, L_y) \quad (\text{E.12}) \end{aligned}$$

where $(\xi_{\ell, m}, \eta_{\ell, m}) \in \mathcal{A}_{\ell, m}$. Summing over all ℓ, m , we obtain

$$\begin{aligned} & \int_{\mathcal{S}} dx dy |U^{(2)}(x - x', y - y')| \rho_{n, n'}(x, y; \Delta \ell) \\ &\leq \frac{eB}{h} \varepsilon^{(2)}(\Delta \ell, n_{\max}, L_x, L_y) \sum_{\ell, m} |U^{(2)}(\xi_{\ell, m} - x', \eta_{\ell, m} - y')| \Delta x \Delta y \quad (\text{E.13}) \end{aligned}$$

We write

$$\begin{aligned} & \sum_{\ell, m} |U^{(2)}(\xi_{\ell, m} - x', \eta_{\ell, m} - y')| \Delta x \Delta y \\ &= \sum_{\ell, m; r_{\ell, m} \leq R} |U^{(2)}(\xi_{\ell, m} - x', \eta_{\ell, m} - y')| \Delta x \Delta y \\ &\quad + \sum_{\ell, m; r_{\ell, m} > R} |U^{(2)}(\xi_{\ell, m} - x', \eta_{\ell, m} - y')| \Delta x \Delta y \quad (\text{E.14}) \end{aligned}$$

with $r_{\ell,m} = \sqrt{\xi_{\ell,m}^2 + \eta_{\ell,m}^2}$ and for a large positive number R . The first sum in the right-hand side is covers to

$$\int_{x^2+y^2 \leq R^2} dx dy |U^{(2)}(x, y)| \quad (\text{E.15})$$

as $L_x, L_y \rightarrow +\infty$ for a fixed large R because $|U^{(2)}|$ is uniformly continuous from the assumption on $U^{(2)}$. The second sum in the right-hand side of (E.14) becomes small for a large R from the assumption (2.8) of $U^{(2)}$ about the decay for a large distance. Combining these observations with (E.4) and (E.13), we obtain the desired bound (E.2). ■

From (E.1), (E.2) and (C.2), we have

$$\begin{aligned} & \sum_{\xi} \sum_{\xi'} \int_S dx dy |U^{(2)}(x-x', y-y')| |\phi_{\xi}^P(x, y)| |\phi_{\xi'}^P(x, y)| \\ & \leq (n_{\max} + 1)^2 \mathcal{C}^{(10)}(U^{(2)}) \left\{ \mathcal{C}^{(4)}(n_{\max}) \sum_{\Delta\ell = -\infty}^{+\infty} \exp \left[- \left(\frac{\pi\ell_B}{L_x} \Delta\ell - n_{\max} \right)^2 \right] \right. \\ & \quad \left. + \mathcal{C}^{(5)}(n_{\max}) \frac{L_x L_y}{2\pi\ell_B^2} \exp \left[- \frac{L_y^2}{32\ell_B^2} \left(1 - \frac{32n_{\max}\ell_B}{L_y} \right)^2 \right] \right\} \quad (\text{E.16}) \end{aligned}$$

Since the sum in the right-hand side can be easily evaluated as

$$\sum_{\Delta\ell = -\infty}^{+\infty} \exp \left[- \left(\frac{\pi\ell_B}{L_x} \Delta\ell - n_{\max} \right)^2 \right] \leq \mathcal{C}^{(11)}(n_{\max}) + \frac{L_x}{\ell_B} \mathcal{C}^{(12)}(n_{\max}) \quad (\text{E.17})$$

we obtain the bound (6.73) with (6.64). Here the constants $\mathcal{C}^{(11)}(n_{\max})$ and $\mathcal{C}^{(12)}(n_{\max})$ depend on the energy cutoff n_{\max} only.

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REFERENCES

1. D. C. Tsui, H. L. Störmer, and A. C. Gossard, *Phys. Rev. Lett.* **48**:1559 (1982).
2. H. L. Störmer, A. M. Chang, D. C. Tsui, J. C. M. Hwang, A. C. Gossard, and W. Wiegmann, *Phys. Rev. Lett.* **50**:1953 (1983).
3. R. E. Prange and S. M. Girvin (eds.), *The Quantum Hall Effect*, 2nd ed. (Springer, 1990).
4. T. Chakraborty and P. Pietiläinen, *The Quantum Hall Effects: Fractional and Integral*, 2nd updated ed. (Springer, 1995).

5. S. Das Sarma and A. Pinczuk (eds.), *New Perspectives in Quantum Hall Effects* (Wiley, 1996).
6. E. H. Lieb, T. Schultz, and D. C. Mattis, *Ann. Phys. (N. Y.)* **16**:407 (1961).
7. I. Affleck and E. H. Lieb, *Lett. Math. Phys.* **12**:57 (1986).
8. I. Affleck, *Phys. Rev. B* **37**:5186 (1988).
9. M. Oshikawa, M. Yamanaka, and I. Affleck, *Phys. Rev. Lett.* **78**:1984 (1997).
10. M. Yamanaka, M. Oshikawa, and I. Affleck, *Phys. Rev. Lett.* **79**:1110 (1997).
11. P. Gagliardini, S. Haas, and T. M. Rice, *Phys. Rev. B* **58**:9603 (1998).
12. K. von Klitzing, G. Dorda, and M. Pepper, *Phys. Rev. Lett.* **45**:494 (1980).
13. S. Kawaji and J. Wakabayashi, in *Physics in High Magnetic Fields*, S. Chikazumi and N. Miura, eds. (Springer-Verlag, Berlin/Heidelberg, 1981), p. 284.
14. T. Koma, *J. Stat. Phys.* **99**:383 (2000).
15. D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, *Phys. Rev. Lett.* **49**:405 (1982); J. E. Avron, R. Seiler, and B. Simon, *Phys. Rev. Lett.* **51**:51 (1983); B. Simon, *Phys. Rev. Lett.* **51**:2167 (1983); M. Kohmoto, *Ann. Phys.* **160**:343 (1985); H. Kunz, *Commun. Math. Phys.* **112**:121 (1987); K. Shizuya, *Phys. Rev.* **45**:11143 (1992); Y. Hatsugai, *Phys. Rev. B* **48**:11851 (1993); J. E. Avron, R. Seiler, and B. Simon, *Commun. Math. Phys.* **159**:399 (1994); J. Bellissard, A. Van Elst, and H. Schulz-Baldes, *J. Math. Phys.* **35**:5373 (1994); K. Ishikawa, N. Maeda, K. Tadaki, and S. Uchiyama, *Phys. Lett. A* **210**:321 (1996); F. Nakano, *J. Math. Sci. Univ. Tokyo* **4**:351 (1997); Calculation of the Hall conductivity by Abel limit, Preprint.
16. R. Tao and Y.-S. Wu, *Phys. Rev. B* **30**:1097 (1984); Q. Niu, D. J. Thouless, and Y.-S. Wu, *Phys. Rev. B* **31**:3372 (1985); J. E. Avron and R. Seiler, *Phys. Rev. Lett.* **54**:259 (1985); J. E. Avron and L. G. Yaffe, *Phys. Rev. Lett.* **56**:2084 (1986).
17. T. Ando, Y. Matsumoto, and Y. Uemura, *J. Phys. Soc. Jpn.* **39**:279 (1975); H. Aoki and T. Ando, *Solid State Commun.* **38**:1079 (1981); R. E. Prange, *Phys. Rev. B* **23**:4802 (1981); R. B. Laughlin, *Phys. Rev. B* **23**:5632 (1981); D. J. Thouless, *J. Phys. C* **14**:3475 (1981); B. I. Halperin, *Phys. Rev. B* **25**:2185 (1982); R. E. Prange and R. Joynt, *Phys. Rev. B* **25**:2943 (1982).
18. T. Koma, Localization and conductance plateaus in quantum Hall systems, in preparation.
19. R. Willett, J. P. Eisenstein, H. L. Störmer, D. C. Tsui, A. C. Gossard, and J. H. English, *Phys. Rev. Lett.* **59**:1776 (1987).
20. B. I. Halperin, *Helv. Phys. Acta.* **56**:75 (1983).
21. J. Zak, *Phys. Rev.* **134**:A1602 and A1607 (1964).
22. O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, 2 volumes (Springer-Verlag, New York, Berlin Heidelberg, 2nd ed., 1987 and 1996).
23. F. D. M. Haldane, *Phys. Rev. Lett.* **55**:2095 (1985).
24. R. Tao and D. J. Thouless, *Phys. Rev. B* **28**:1142 (1983); R. Tao, *Phys. Rev. B* **29**:636 (1984).
25. R. B. Laughlin, *Phys. Rev. Lett.* **50**:1359 (1983); K. Ishikawa and N. Maeda, *Prog. Theor. Phys.* **97**:507 (1997).
26. T. Matsui, *Commun. Math. Phys.* **182**:723 (1996).
27. T. Koma and H. Tasaki, *J. Stat. Phys.* **76**:745 (1994).
28. D. Yoshioka, B. I. Halperin, and P. A. Lee, *Phys. Rev. Lett.* **50**:1219 (1983); *Surf. Sci.* **142**:155 (1984).
29. D. Yoshioka, *Phys. Rev. B* **29**:6833 (1984); W. P. Su, *Phys. Rev. B* **30**:1069 (1984); R. Tao and Y.-S. Wu, *Phys. Rev. B* **30**:1097 (1984).
30. H. Tasaki, private communication.
31. O. Bratteli, A. Kishimoto, and D. Robinson, *Commun. Math. Phys.* **64**:41 (1978).